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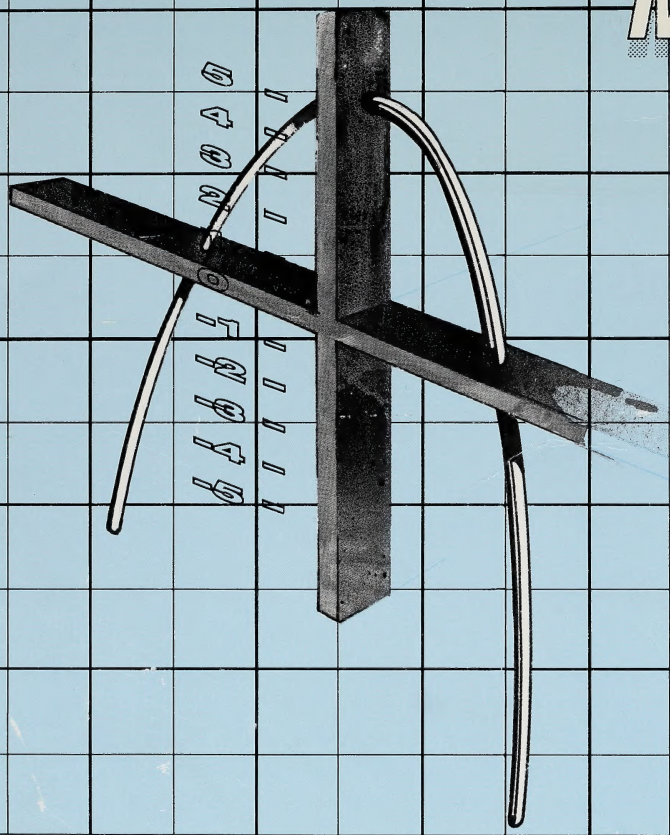
MATHEMATICS 30

PERMUTATIONS AND COMBINATIONS
UNIT 7



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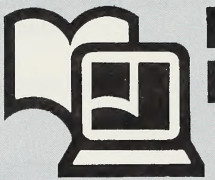

Distance Learning

You have chosen an alternate form of learning that allows you to work at your own pace. You will be responsible for your own schedule, for disciplining yourself to study the units thoroughly, and for completing your units regularly. We wish you much success and enjoyment in your studies.

Mathematics 30 Student Module Unit 7 Permutations and Combinations Alberta Correspondence School ISBN No. 0-7741-0056-7

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General Information

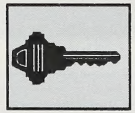
This information explains the basic layout of each booklet.

- **What You Already Know** and **Review** are to help you look back at what you have previously studied. The questions are to jog your memory and to prepare you for the learning that is going to happen in this Unit.
- As you begin each **Topic**, spend a little time looking over the components. Doing this will give you a preview of what will be covered in the topic and will set your mind in the direction of learning.
- **Exploring the Topic** includes the objectives, concept development and activities for each objective.Use your own papers to arrive at the answers in the activities.
- **Extra Help** reviews the topic. If you had any difficulty with **Exploring the Topic**, you may find this part helpful.
- **Extensions** gives you the opportunity to take the topic one step further.
- To summarize what you have learned, and to find instructions on doing the unit assignment, turn to the **Unit Summary** at the end of the unit.
- The **Appendices** include the solutions to **Activities (Appendix A)** and any other charts, tables, etc. which may be referred to in the topics **(Appendix B, etc.)**.

Visual Cues

Visual cues are pictures that are used to identify important areas of the material. They are found throughout the booklet.

An explanation of what they mean is written beside each visual cue.

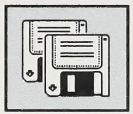


Key Idea
- flagging important ideas

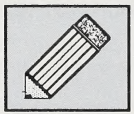
Audiotape
- learning by listening to an audiotape

What You Already Know
- reviewing what you already know

Another View
- exploring different perspectives



Computer Software
- learning by using computer software

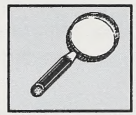


Review
- studying previous concepts

Solutions
- correcting the activities

Videotape
- learning by viewing a videotape

Introduction
- introducing the unit



Extra Help
- providing additional study

Print Pathway
- choosing a print alternative

What Lies Ahead
- previewing the unit

Extensions
- going on with the topic

Calculator
- using your calculator

Exploring the Topic
- actively learning new concepts

What You Have Learned
- summarizing what you have learned

Mathematics 30

Course Overview

Mathematics 30 contains 7 units. Beside each unit is a percentage that indicates what the unit is worth in relation to the rest of the course. The units and their percentages are listed below. You will be studying the unit that is shaded.

Unit 1 Polynomial Functions	12%
Unit 2 Logarithms	10%
Unit 3 Sequences, Series, Limits	13%
Unit 4 Trigonometry	15%
Unit 5 Quadratic Relations	20%
Unit 6 Statistics	18%
Unit 7 **Permutations and Combinations**	**12%**
	100%

Unit Assessment

After completing the unit you will be given a mark based totally on a unit assignment. This assignment will be found in the Assignment Booklet.

Unit Assignment - 100%

If you are working on a CML terminal your teacher will determine what this assessment will be. It may be

Unit assignment - 50%
Supervised unit test - 50%

Introduction to Permutations and Combinations

This unit covers topics dealing with Permutations and Combinations. Each topic contains explanations, examples, and activities to assist you in understanding permutations and combinations. If you find you are having difficulty with the explanations and the way the material is presented, there is a section called **Extra Help**. If you would like to extend your knowledge of the topic, there is a section called **Extensions**.

You can evaluate your understanding of each topic by working through the activities. Answers are found in the Solutions in the **Appendix**. In several cases there is more than one way to do the question.

Unit 7 Permutations and Combinations

Contents at a Glance

Value	**Permutations and Combinations**	3
	What You Already Know	5
	Review	5
45%	**Topic 1:** Permutations	6
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 1	
55%	**Topic 2:** Combinations	35
	• Introduction • Extra Help • What Lies Ahead • Extensions • Exploring Topic 2	
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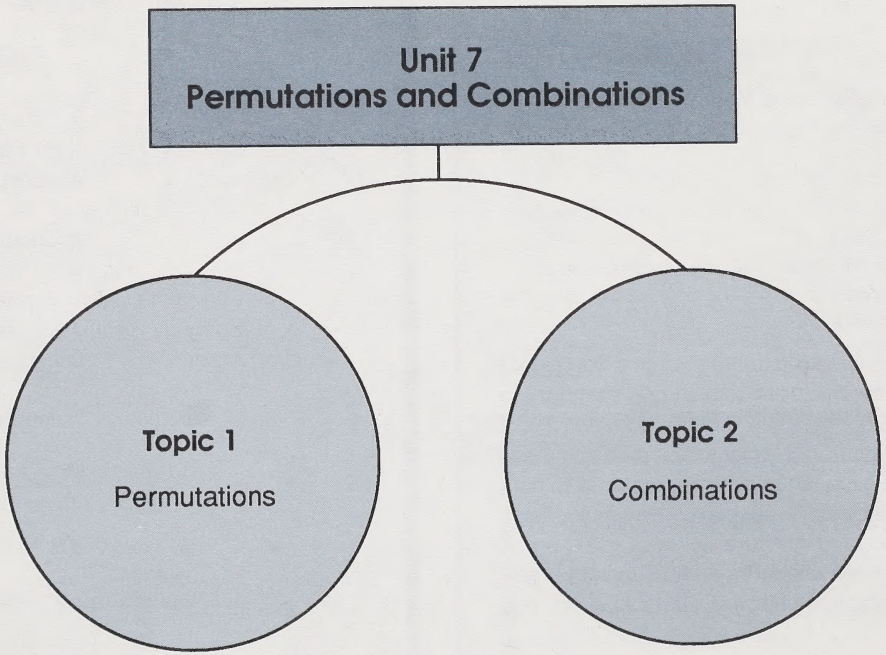
Permutations and Combinations

Did you ever forget your combination lock number? Did you find it fun (or frustrating) when you tried to open it? Did you spend much time this morning to decide what colour shirt and what colour pants you would like to wear? When you faced this kind of problem, did you know how many possible ways you could solve your problem? Permutation and combination problems happen in our daily life. In this unit, these two kinds of problems will be discussed and they will also be used in probability and the binomial theorem.

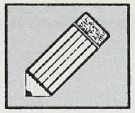

Unit 7
Permutations and Combinations
Topic 1
Permutations
Topic 2
Combinations

What You Already Know

In Mathematics 20, you learned probability. Do you remember the following?

- Probability $= \dfrac{\text{number of favourable cases}}{\text{number of possible cases}}$
- For independent events, $P(A \text{ and } B) = P(A) \times P(B)$.
- For dependent events, $P(A \text{ and } B) = P(A) \times P(B \,/\, A)$.
- Addition Law $= P(A \text{ or } B) = P(A) + P(B) - P(A \cap B)$
- For mutually exclusive events, $P(A \text{ or } B) = P(A) + P(B)$.

Now that you have looked at material that you studied previously, go to the **Review** to confirm your understanding of this material. If you do not remember all of this material, please return to the probability unit of the Mathematics 20 course.



Review

Now try the following questions.

1. Two coins are tossed. What is the probability that both coins land heads up?

2. One bag contains six pennies and three dimes. A second bag contains seven pennies and five dimes. One coin is drawn from each bag. What is the probability that both are dimes?

3. Two coins are drawn from a bag, one after the other. The bag contains twelve dimes and seven quarters. The first coin drawn will not be replaced before the second coin is drawn. What is the probability that both coins are quarters?

4. Consider the experiment of rolling a red die and a blue die. What is the probability of obtaining a 5 on the red die or a 4 on the blue die?

5. Consider the experiment of rolling a die. What is the probability of obtaining a 3 or a 1?

Now go to the **Review** solutions in the **Appendix**.

Topic 1 Permutations

Introduction

When you try to arrange your books on a bookshelf, do you know how many different ways you can arrange them? The technique used to find the answer for this kind of question comes from the branch of mathematics called combinatorics. The fundamental counting principle and permutations are part of combinatorial analysis. In this topic, you are going to solve permutation problems and use permutation to solve problems in probability.

What Lies Ahead

Throughout this topic you will learn to

1. solve problems using the fundamental counting principle
2. define and use ${}_nP_n$ and $n!$
3. define and use ${}_nP_r$; also, solve linear permutations with repetitions and permutations with like elements
4. solve problems involving circular permutations and permutations with reflections
5. solve probability questions which involve the use of permutations

Now that you know what to expect, turn the page to begin your study of permutations.

Exploring Topic 1

Activity 1

Solve problems using the fundamental counting principle.

If there are five candidates for student union chairperson and four for secretary, the two positions may be filled in $5 \times 4 = 20$ ways. In general, if there are r different ways to do one thing and s different ways to do another, there are rs ways to do these two things in succession. This reasoning is called the **fundamental counting principle**.

The fundamental counting principle can be extended when more actions are involved. If the first thing can be done in a ways, the second thing in b ways, the third thing in c ways . . . and so on, then the number of ways of doing all these things in succession is $a \times b \times c$ . . . ways. The following example is another way to illustrate this reasoning.

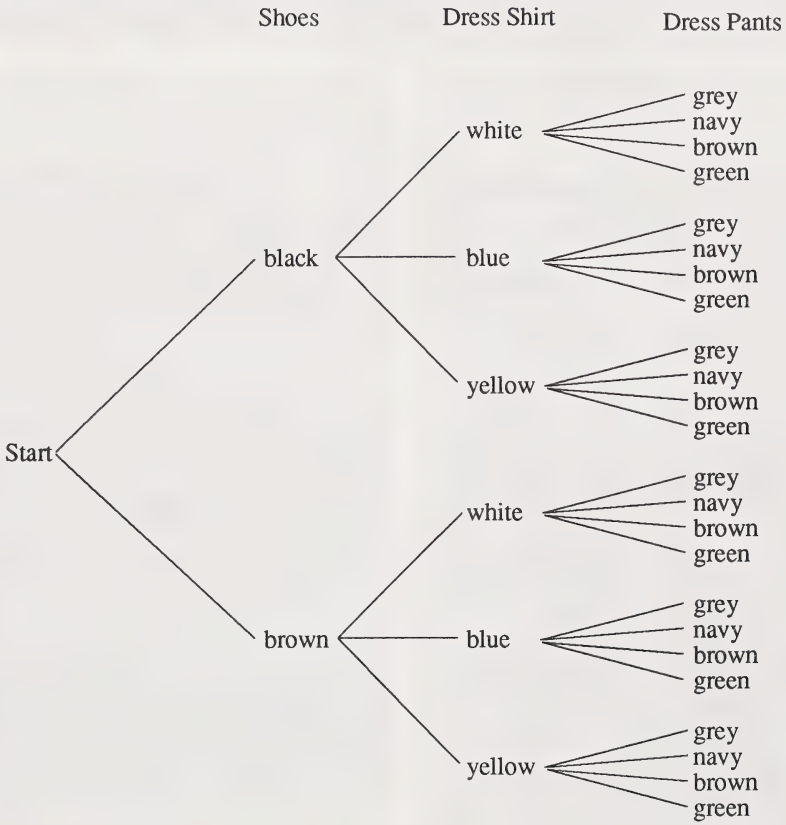
Example 1

John has two pairs of shoes, one black and the other brown. He has three dress shirts in white, blue, and yellow. He also has dress pants in grey, navy, brown, and green. In how many different ways can he dress himself?

Solution:

A pair of shoes can be selected in two ways.
A dress shirt can be selected in three ways.
A pair of dress pants can be selected in four ways.

A tree diagram shown below illustrates the process.



By the fundamental counting principle, he can dress himself in $2 \times 3 \times 4 = 24$ ways.

Each of the next two examples shows a different situation.

Example 2

A mathematics test consists of ten multiple-choice questions. Each question has three answers. How many different sets of answers can be chosen?

Solution:

There are three choices for each question.
The total number of choices is as follows:

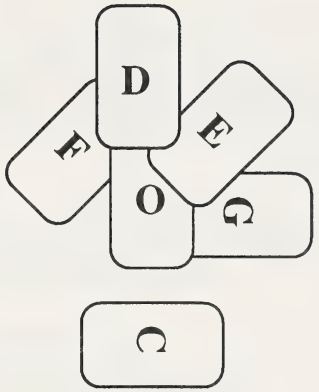
$$3\times3\times3\times3\times3\times3\times3\times3\times3\times3 = 3^{10}$$
$$= 59\,049 \text{ ways}$$

PART A: RESPONSE PAGE

A	1.	B	6.
C	2.	B	7.
B	3.	C	8.
A	4.	C	9.
A	5.	C	10.

Example 3

How many different three-letter "words" can be formed from the letters **C**, **D**, **E**, **F**, **G**, and **O** if no letter is repeated?



Solution:

The first letter can be chosen from any one of the six letters. Having filled this position, the second letter is chosen from the remaining five letters, and consequently, the third letter is chosen from the remaining four letters.

The number of three-letter "words" is $6 \times 5 \times 4 = 120$.

Do the following questions.

1. How many three-digit numbers less than 400 can be formed from the digits 2, 3, 5, 7, and 8? (Digits can be repeated.) Solve this problem by filling in the blanks.

   a. Can the first digit be 5, 7, or 8? _____

   b. The first digit can be chosen in _____ ways.

   c. The second digit can be chosen in _____ ways.

   d. The third digit can be chosen in _____ ways.

   e. Apply the fundamental counting principle. The number of three-digit numbers is _____.

2. The hockey stadium has fifteen gates; seven are on the north side and eight are on the south side.

   a. In how many ways can Peter enter by a south gate and leave by a north gate?

   b. In how many ways can Peter enter and leave the stadium?

3. Show with a diagram the number of ways of travelling from Town *A* to City *C* via Village *B*, if there are two roads (Highways 14 and 15) connecting Town *A* and Village *B*, and there are four roads (Highways 16, 17, 18, and 19) connecting Village *B* and City *C*.

4. How many three-digit odd numbers between 200 and 600 can be formed from the digits 1, 2, 4, 7, and 9? (Digits cannot be repeated.)

5. How many different licence plates can be made using three letters followed by three digits from the digits 1, 2, 3, . . . , 8, 9? (Repetition of letters and digits is not allowed.)

6. In how many ways can five letters be mailed if there are two mailboxes available?

For solutions to **Activity 1**, turn to the **Appendix, Topic 1**.

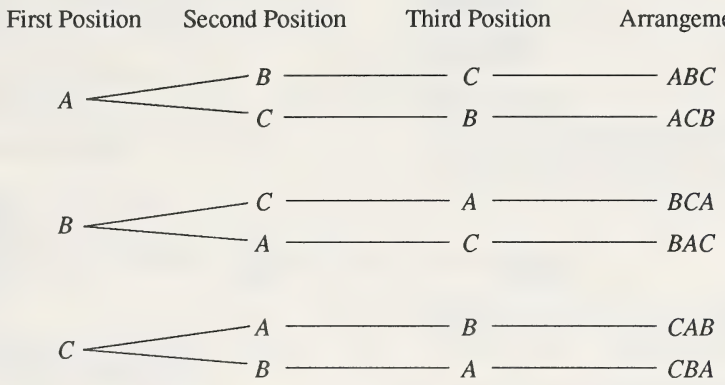
Activity 2

Define and use ${}_nP_n$ and $n!$.

In the last activity, the fundamental counting principle was used to find the number of ways in which actions can be performed. In this activity, more attention will be paid to the arrangements of objects where order must be considered. An arrangement of things along a straight line in a definite order is called a **linear permutation**.

permutation = arrangement

Three boys, *A*, *B*, and *C*, are lined up to buy ice cream. How many different ways can they be arranged in a line? The following tree diagram shows the possible arrangements.



From the tree diagram, you can see that *A*, *B*, and *C* can be arranged in six ways. We can solve this problem in another way. Think of the three positions as three empty boxes, $\square\square\square$, that have to be filled.

There are three ways to fill the first position (any one of the three boys). Having filled the first position, there are two ways to fill the second position (only two boys left). Having filled the second position, there is only one way to fill the last position (only one boy left).

3	2	1

By the fundamental counting principle, the total number of arrangements of three boys in a line is $3 \times 2 \times 1 = 6$.

Look at another example.

Example 4

How many different seven-digit numbers can be made from the digits 1 through 7 with no repetition?

Solution:

Let the number be represented by seven boxes.

The first digit could be any one of the seven digits, the second digit could be any one of the six remaining digits, and so on. This could be shown by filling in the numbers in the boxes.

7	6	5	4	3	2	1

Therefore, $7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 5040$ numbers can be made from these seven digits.

The above example represents a permutation of seven digits taken seven at a time. There is a special symbol used to designate the number of permutations.

$_nP_n$

This symbol is used to represent the number of permutations of n things taken n at a time. In Example 4, there are seven digits taken all seven at a time. The symbol used to represent the number of permutations is $_7P_7$. Therefore, $_7P_7 = 5040$.

If there are n objects, the number of permutations of all of the objects will be $_nP_n = (n)(n-1)(n-2)(n-3) \ldots 3 \times 2 \times 1$.

Not every permutation is a linear permutation.

In combinatoric questions, multiplication of many consecutive numbers is very common. In order to avoid writing all the numbers, a new notation is used. ***n*!** is read as ***n* factorial** and it represents the product of the **first** n natural numbers. For example,

$$1! = 1$$
$$2! = 2 \times 1$$
$$3! = 3 \times 2 \times 1$$
$$4! = 4 \times 3 \times 2 \times 1$$
$$\ldots\ldots\ldots$$
$$n! = n(n-1)(n-2)\ldots 3 \times 2 \times 1$$

In Example 4, ${}_7P_7 = 7!$.

How do you calculate 12!? It means a lot of multiplications. A scientific calculator can make things much easier for you. Turn on your calculator and follow the two steps:

Step 1: Enter the number.

Step 2: Press the INV key followed by the $x!$ key. The number displayed is the answer you are looking for.

For example, if you want to calculate 12!, you enter the number 12, then press the INV key followed by the $x!$ key. The answer 479 001 600 will appear on screen.

It is important that you know how to simplify factorials. Look at the following examples.

Example 5

Simplify $\frac{9!}{4!}$.

Solution:

$$\frac{9!}{4!} = \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{4!}$$
$$= \frac{9 \times 8 \times 7 \times 6 \times 5 \times 4!}{4!}$$
$$= 9 \times 8 \times 7 \times 6 \times 5$$
$$= 15\,120$$

Note that $n!$ can be written in a variety of equivalent forms.

$$n! = n(n-1)!$$
$$= n(n-1)(n-2)!$$
$$= n(n-1)(n-2)(n-3)!$$
$$\ldots\ldots\ldots\ldots\ldots$$

When a factorial is denoted by a variable, it is necessary to change it to one of these equivalent forms so that you can reduce a rational expression with factorials in the numerator and denominator. Look at the following examples.

Example 6

Simplify $\frac{n!}{(n-3)!}$.

Solution:

$$\frac{n!}{(n-3)!} = \frac{n(n-1)(n-2)(n-3)!}{(n-3)!}$$
$$= n(n-1)(n-2)$$
$$= n^3 - 3n^2 + 2n$$

Example 7

Simplify $\frac{(n-2r+3)!}{(n-2r)!}$.

Solution:

$$\frac{(n-2r+3)!}{(n-2r)!}$$

$$= \frac{(n-2r+3)(n-2r+2)(n-2r+1)(n-2r)!}{(n-2r)!}$$

$$= (n-2r+3)(n-2r+2)(n-2r+1)$$

Some word problems in arrangements are not that straightforward. Look at the next example.

Example 8

A bookshelf contains five different algebra books and seven different physics books. How many different ways can these books be arranged if the algebra books are to be kept together?

Solution:

There are twelve books on the shelf.
The five algebra books can be arranged in 5! ways.
The algebra books are considered as one unit, so there are eight items (1 algebra unit + 7 physics books) to be arranged. The number of arrangements is ${}_8P_8$.

$\therefore$ The total number of arrangements

$$= 5! \times {}_8P_8$$
$$= 5! \times 8!$$
$$= 120 \times 40\ 320$$
$$= 4\ 838\ 400 \text{ ways}$$

$(n - 2r + 2)$ is one less than $(n - 2r + 3)$.

$(n - 2r + 1)$ is one less than $(n - 2r + 2)$.

Do at least the first six questions.

1. In how many ways can the letters in the word ALBERTA be arranged?

2. In how many different orders can Minal read five different books?

3. Find the value of $\frac{10!}{7!2!}$.

4. Find the value of $\frac{4!}{5!2!}+\frac{6!}{3!2!}$.

5. Express $(k-2)(k-3)!$ as a single factorial.

6. Simplify $\frac{(n+r-2)!}{(n+r-5)!}$.

7. Express $(m+3)(m+2)!$ as a single factorial.

8. Solve for n: $\frac{(n-3)!}{(n-4)!}=7$.

9. How many six-letter "words" can be formed using all the letters **N, U, M, B, E, R**, if the first letter must be a vowel?

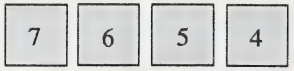
For solutions to **Activity 2**, turn to the **Appendix, Topic 1**.

Activity 3

Define and use ${}_nP_r$; also, solve linear permutations with repetitions and permutations with like elements.

It was defined in Activity 2 that ${}_nP_n = n!$ is the number of linear permutations of n objects taken n at a time. What is the number of linear permutations of n objects taken r at a time? For example, how many different four-digit numbers can be made from the digits 1 through 7 with no repetitions? In Example 4, seven boxes were used to house the seven digits taken at a time. Now the number to be made has four digits. Only four boxes are needed.

The first digit could be any one of seven digits, the second digit could be any one of the six remaining digits, the third digit could be any one of the five remaining digits, and the fourth digit could be any one of the four remaining digits.



By the fundamental counting principle, the total number of four-digit numbers is $7 \times 6 \times 5 \times 4 = 840$.

In general, if there are n objects taken r at a time, then the first object must be selected from a total of n objects, the second object must be selected from $(n-1)$ remaining objects, the third one must be selected from $(n-2)$ remaining objects, and the nth object must be selected from $(n-r+1)$ remaining objects. There is a symbol used to denote the number of permutations of n objects taken r at a time:

$${}_nP_r$$

Therefore, ${}_nP_r = n(n-1)(n-2)\ldots(n-r+1)$.

Since $n(n-1)(n-2)\ldots(n-r+1)(n-r)! = n!$,

$${}_nP_r(n-r)! = n!.$$

$${}_nP_r = \frac{n!}{(n-r)!}$$

According to this formula, ${}_nP_n = \frac{n!}{(n-n)!}$

$$= \frac{n!}{0!}.$$

In Activity 2, ${}_nP_n$ is defined as $n!$. For this formula ${}_nP_r = \frac{n!}{(n-r)!}$ to hold for $r = n$, 0! must be defined as 1 so that ${}_nP_n = \frac{n!}{0!} = \frac{n!}{1} = n!$.

Since 1! = 1, $\mathbf{0! = 1! = 1}$.

Look at the following examples.

Example 9

At a parking lot, five cars are to be parked side by side along the wall. If there are seven cars, how many different ways can this be done?

Solution:

Cars are to be parked side by side along the wall; therefore, it is a linear permutation problem.

This problem can be solved by using the formula ${}_nP_r = \frac{n!}{(n-r)!}$, where $n = 7$ and $r = 5$.

$${}_7P_5 = \frac{7!}{(7-5)!}$$

$$= \frac{7\times6\times5\times4\times3\times2!}{2!}$$

$$= 7\times6\times5\times4\times3$$

$$= 2520 \text{ ways}$$

Note: $n \geq r$

Make sure that you remember this formula:

$${}_nP_r = \frac{n!}{(n-r)!}$$

If you use five boxes to represent the parking spaces, the first one can be used to park any one of the seven cars.
The second one can be used to park any one of the six remaining cars.
The third one can be used to park any one of the five remaining cars.
The fourth one can be used to park any one of the four remaining cars.
The fifth one can be used to park any one of the three remaining cars.

By the fundamental counting principle, the number of possible ways is $7 \times 6 \times 5 \times 4 \times 3 = 2520$ ways.

Sometimes you have to add, subtract, or multiply arrangements. It is important that you can do those operations.

Example 10

Evaluate ${}_3P_1 \times {}_6P_4$.

Solution:

$$\begin{aligned} {}_3P_1 \times {}_6P_4 &= \frac{3!}{(3-1)!} \times \frac{6!}{(6-4)!} \\ &= \frac{3!}{2!} \times \frac{6!}{2!} \\ &= (3) \times (6 \times 5 \times 4 \times 3) \\ &= 1080 \end{aligned}$$

Example 11

Evaluate ${}_5P_4 + {}_7P_1 - {}_3P_2$.

Solution:

$$\begin{aligned} {}_5P_4 + {}_7P_1 - {}_3P_2 &= \frac{5!}{(5-4)!} + \frac{7!}{(7-1)!} - \frac{3!}{(3-2)!} \\ &= 120 + 7 - 6 \\ &= 121 \end{aligned}$$

The next three examples are harder questions.

Example 12

Six signal flags of different colour can be used to send different signals. If no more than four flags can be used at any one time, how many different signals can be sent by flag(s) flown from a flagpole? (No flags is not considered as a signal.)

Solution:

If you choose any four flags to form a signal, the number of signals is ${}_6P_4$.

If you choose any three flags to form a signal, the number of signals is ${}_6P_3$.

If you choose any two flags to form a signal, the number of signals is ${}_6P_2$.

If you choose any one flag to form a signal, the number of signals is ${}_6P_1$.

The total number of signals $= {}_6P_4 + {}_6P_3 + {}_6P_2 + {}_6P_1$

$$= \frac{6!}{(6-4)!} + \frac{6!}{(6-3)!} + \frac{6!}{(6-2)!} + \frac{6!}{(6-1)!}$$

$$= \frac{6!}{2!} + \frac{6!}{3!} + \frac{6!}{4!} + \frac{6!}{5!}$$

$$= (6\times5\times4\times3) + (6\times5\times4) + (6\times5) + (6)$$

$$= 360 + 120 + 30 + 6$$

$$= 516 \text{ ways}$$

Example 13

Determine the value of n which makes the following statement true.
${}_nP_6 = 8 \times {}_{n-1}P_5$

Solution:

$${}_nP_6 = \frac{n!}{(n-6)!}$$

$${}_{n-1}P_5 = \frac{(n-1)!}{(n-1-5)!}$$

$$= \frac{(n-1)!}{(n-6)!}$$

If $\frac{n!}{(n-6)!} = 8 \times \frac{(n-1)!}{(n-6)!}$,

then $n! = 8 \times (n-1)!$.

$$\frac{n!}{(n-1)!} = 8$$

$$\frac{n(n-1)!}{(n-1)!} = 8$$

$$n = 8$$

Example 14

There are two girls, five boys, and five chairs in a row. It is required that five persons be seated in a row such that the two girls occupy the first two seats. How many different ways can this be done?

Solution:

The two girls are considered as one unit. There are ${}_2P_2 = 2$ ways to seat these two girls.

The boys may be seated in ${}_5P_3$ ways because there are three chairs for five boys.

The total number of ways $= {}_2P_2 \times {}_5P_3$

$$= 2 \times \frac{5!}{(5-3)!}$$

$$= 2 \times (5\times4\times3)$$

$$= 120 \text{ ways}$$

Sometimes the objects (e.g. numbers) can be used more than once in an arrangement. In that case, the fundamental counting principle can provide the best explanation.

Example 15

How many different five-digit credit card numbers can be formed if the first digit cannot be zero?

Solution:

There are ten digits: 0, 1, 2, 3, 4, 5, 6, 7, 8, and 9.
The first digit cannot be zero. Therefore, there are nine ways to choose the first digit.
The other digits can be any one of the ten digits.
If five boxes are used to house these five digits, the first box can be filled by any one of the nine digits.
There are ten ways to fill each of the other four boxes.

9	10	10	10	10

The number of five-digit numbers is $9 \times 10 \times 10 \times 10 \times 10 = 90\,000$.

The next example has more restrictions:

Example 16

How many six-digit odd numbers less than 500 000 can be formed from the digits 3, 4, 5, 6, 7, and 8?

Solution:

The first digit can be 3 or 4.
The last digit must be an odd number. (3 odd numbers)
The other digits can be any one of the six digits.

Use six boxes to house the numbers.

2	6	6	6	6	3

The number of arrangements is $2 \times 6 \times 6 \times 6 \times 6 \times 3 = 7776$.

If some of the objects you want to arrange are identical, how do you determine the number of permutations? For example, how many arrangements can be formed by using the letters of the word SEE?

If the letters were different, there would be $3! = 6$ ways to arrange them. The problem is that the two **E**'s are really identical and indistinguishable. If you pretend that the two **E**'s are different and use a capital letter **E** and a small letter **e** to denote the two of them, the possible arrangements are as follows:

SEe	SeE
ESe	eSE
EeS	eES

Since the **E**'s are really identical, there are actually three different permutations: SEe, ESe, and EeS. For every position of the letter **S**, there are $2! = 2$ ways to arrange the **E**'s. Therefore, the total number of arrangements is $\frac{3!}{2!} = \frac{3 \times 2 \times 1}{2 \times 1} = 3$ ways.

In general, the number of permutations of n objects taken all at a time, of which u are alike, v are the same of another kind, w are the same of another kind, etc., is as follows:

$$P = \frac{n!}{u!v!w!\ldots}, \text{ where } u+v+w+\ldots = n$$

Look at the following example.

Example 17

How many distinguishable permutations can be made from all the letters of the word REFERENCE?

Solution:

There are two **R**'s and four **E**'s.

The number of permutations

$$= \frac{9!}{2!4!1!1!1!}$$

$$= \frac{9\times8\times7\times6\times5\times4!}{2!4!}$$

$$= 9\times8\times7\times3\times5$$

$$= 7560 \text{ ways}$$

Sometimes the objects do not have to be identical, but you have to treat them as if they were alike.

Example 18

How many distinguishable permutations can be made from all the letters of the word REACH if the consonants must be in alphabetical order?

Solution:

If the order of the consonants cannot be changed, they can be treated as if they were alike. The three consonants are **C**, **H**, and **R**.

$$P = \frac{5!}{3!1!1!}$$

$$= 5\times4$$

$$= 20 \text{ ways}$$

Do at least the first seven questions.

1. In how many ways can the first four positions be taken in a horse race of ten horses excluding any ties?

2. Evaluate ${}_4P_3 + {}_5P_2$.

3. How many different numbers can be formed from the digits 1, 2, 3, 4, and 5 if no more than four digits can be used and no repetition is allowed?

4. Determine the value of n which makes the following statement true.

   $\left(\frac{1}{2}\right){}_nP_7 = {}_{n-2}P_5,\ n \in N$

5. In how many ways can five books be arranged on a bookshelf if there are three different chemistry books and two different physics books, and it is also required that the two physics books are together and on the shelf?

REFERENCE has two **R**'s, four **E**'s, one **F**, one **N**, and one **C**.
$2 + 4 + 1 + 1 + 1 = 9$

6. How many distinguishable permutations can be made from all the letters of the word PEERLESS?

7. How many serial numbers can be formed using two letters followed by four digits? If no restriction is mentioned in the question, it usually means that repetitions are allowed for both letters and digits.

8. How many different arrangements can be made by four different marbles and four different colour beads in a row, if no two marbles and no two beads are to be placed next to each other?

9. A student has two different biology books, three different chemistry books, and five different physics books. In how many ways can these books be arranged on a bookshelf so that the biology books stand together, the chemistry books stand together, and the physics books stand together?

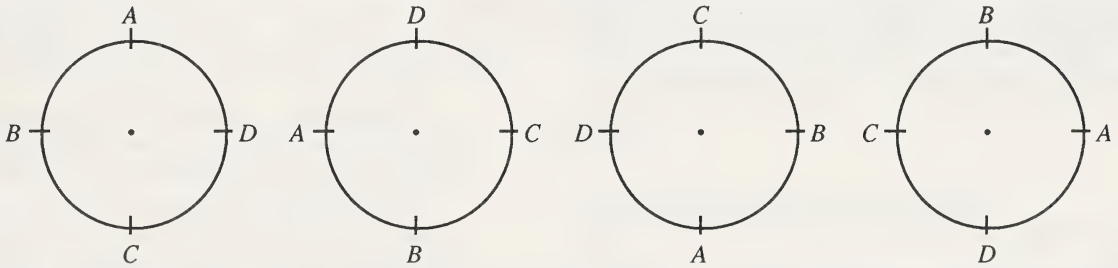
10. A teacher wants to distribute three copies of a chemistry book and five copies of a physics book to eight students, and each student is to receive one book. In how many different ways can this be done?

For solutions to **Activity 3**, turn to the **Appendix, Topic 1**.

Activity 4

Solve problems involving circular permutations and permutations with reflections.

In the previous activity, you studied linear permutations. In this activity, you are going to study circular permutations. Imagine a merry-go-round of four persons, *A*, *B*, *C*, and *D*, equally spaced. The four arrangements illustrated below would be equivalent.



Only when the position of a person relative to another one is changed, is a different arrangement made. In order to find the number of distinguishable arrangements, a person is assigned a position in the circle. Then the number of distinguishable arrangements is the number of arrangements of the remaining $(4-1)=3$ persons as if they were in a straight line. For this merry-go-round of four persons, there are 3! arrangements.

In general, the number of ways of arranging n objects around a circle is $(n-1)!$ ways.

Look at the following examples.

Example 19

In how many different ways can five persons be arranged in a circle?

Solution:

This is a circular permutation problem. You have to fix the position of one of the five persons.
Let one of them be seated anywhere.
The number of ways of arranging five persons in a circle $=(5-1)!$

$=4!$

$=24$ ways

Example 20

In how many ways can seven people be seated at a round table if three of them must always sit together?

Solution:

These three particular persons can be arranged in 3! ways.
Consider these three particular persons as one person.
The remaining four persons plus this one person make up five persons.
Five persons can be seated at a round table in $(5-1)!$ ways.
The required number of ways $=3!\times(5-1)!$

$=(3\times2\times1)(4\times3\times2\times1)$

$=6\times24$

$=144$ ways

Example 21

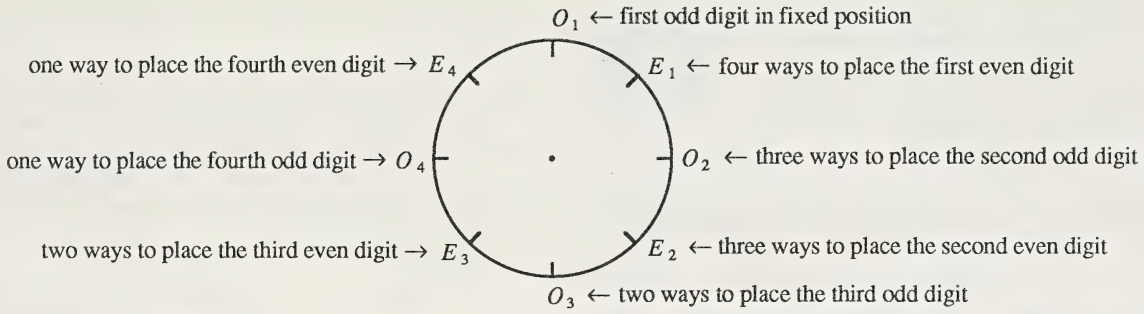
In how many ways can the digits 1, 2, 3, 4, 5, 6, 7, and 8 be placed in a circle if each even digit is to be between two odd digits?

Solution:

Assume that the odd digits are to be placed first. One position is assigned to an odd digit, and the other three odd digits can be arranged in 3! ways.
Now there are four positions for the four even digits. They can be arranged in 4! ways.
The total number of circular permutations $=3!\times4!$

$=144$ ways

Look at this question from a different perspective. In the following diagram, let O represent odd digits and E represent even digits.


O_1 ← first odd digit in fixed position
one way to place the fourth even digit → E_4
E_1 ← four ways to place the first even digit
one way to place the fourth odd digit → O_4
O_2 ← three ways to place the second odd digit
two ways to place the third even digit → E_3
E_2 ← three ways to place the second even digit
O_3 ← two ways to place the third odd digit

The total number of circular permutations is $1 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 144$ ways.

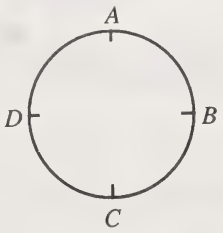
Do the following questions.

1. In how many different ways can nine folk dancers form a circle?
2. A florist wants to use ten different flowers and two identical roses to make a circular arrangement. In how many ways can this be done?
3. Five boys and five girls sit at a round table. Boys and girls must alternate. In how many different ways can this be done?
4. Seven different marbles are placed in a circle. How many different arrangements can be made if three of them must be placed in a position adjacent to each other?

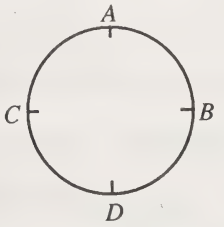
For solutions to **Activity 4**, turn to the **Appendix, Topic 1**.

You have just learned that n objects can be arranged in a circle in $(n-1)!$ ways. Consider the example of arranging four objects (label them A, B, C, and D) in a circle. From your previous work, you know that four objects can be arranged in a circle in $(4-1)!$ or six ways. The six arrangements are shown below.

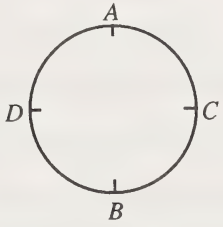
Arrangement 1



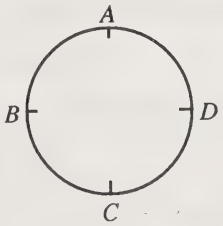
Arrangement 2



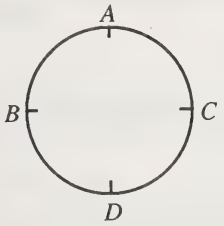
Arrangement 3



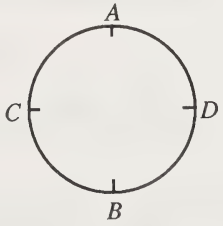
Arrangement 4



Arrangement 5

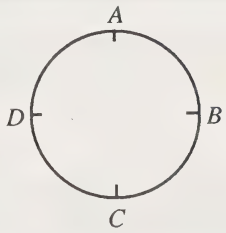


Arrangement 6

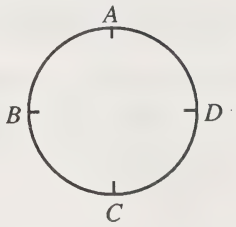


If these arrangements can be turned over as in the case of a necklace or a key chain, Arrangements 1 and 4 above are identical since Arrangement 4 can be obtained by flipping the ring in Arrangement 1 about a vertical line AC, without repositioning the objects on the ring. Arrangement 4 is also the mirror reflection of Arrangement 1.

Arrangement 1



Arrangement 4



Similarly, Arrangements 2 and 5 are identical if you are dealing with a ring that can be turned over. Also, Arrangements 3 and 6 are identical. Therefore, four objects can be arranged in a ring (that can be turned over) in only three ways. In other words, the number of arrangements is half as many as before.

This kind of arrangement like a necklace or a key chain is called a circular permutation with reflections or a **ring permutation**.

The number of ways of arranging n objects in a ring that can be turned over (e.g. a necklace or a key chain) is as follows:

$$\frac{(n-1)!}{2}$$

Example 22

In how many ways can nine different charms on a charm bracelet be arranged?

Solution:

$$\text{The number of ways} = \frac{(n-1)!}{2}$$
$$= \frac{(9-1)!}{2}$$
$$= \frac{8!}{2}$$
$$= 20\,160 \text{ ways}$$

Example 23

A key chain has seven different keys. If two of them are attached together, how many different arrangements can be made?

Solution:

The two keys together are considered as one group and can be arranged in 2! = 2 ways.
The remaining five keys and this one group can be arranged in $\frac{(6-1)!}{2}$ ways.
Therefore, the total number of arrangements $= 2 \times \frac{(6-1)!}{2}$
$= 5!$
$= 120$ ways

Do the following questions.

5. In how many ways can five keys be arranged on a key chain?

6. How many different bracelets can be made by stringing five red beads and four white beads if beads of the same colour are identical?

For solutions to **Activity 4**, turn to the **Appendix, Topic 1**.

Activity 5

Solve probability questions which involve the use of permutations.

You learned probability in Mathematics 20.

$$\text{Probability} = \frac{\text{number of favourable cases}}{\text{number of possible cases}} = \frac{n(A)}{n(S)}$$

Many questions in probability involve permutations. If the number of possible cases represents the number of arrangements, you have to use permutations to determine the number of cases. Look at the following examples.

Example 24

Ten people line up at a theatre. They are of different heights. What is the probability that they form a queue in descending order of height?

Solution:

If these ten people line up randomly, then the number of possible queues is the number of arrangements of these ten people.

${}_{10}P_{10} = 10!$

Only one queue has all ten people in descending order of height; therefore, the number of favourable cases is one.

The probability that the ten people line up in descending order of height is as follows:

$$P = \frac{1}{10!}$$

$$= \frac{1}{3\,628\,800}$$

Example 25

There are twelve bicycles in a race. Horst, Paul, and Ken are members of the same team. What is the probability that they would finish in the top three?

Solution:

The total possible cases consist of any group of three persons to finish in the top three.

Hence, $n(S) = {}_{12}P_3$

$$= \frac{12!}{(12-3)!}$$

$$= \frac{12!}{9!}$$

$$= 12 \times 11 \times 10$$

$$= 1320.$$

Use ${}_nP_r$ to denote permutation. Use P or $P(E)$ to denote probability.

The number of favourable cases is the number of cases that Horst, Paul, and Ken would finish in the top three.

Hence, $n(A) = {}_3P_3$

$= 3!$

$= 6.$

The probability is $P = \frac{6}{1320}$

$= \frac{1}{220}.$

Do the following questions.

1. Each number from 1 to 7 is written on a piece of paper and placed in a box. All seven numbers are drawn at random from the box one at a time. What is the probability that the numbers drawn are in descending order?

2. There are eight swimmers in a competition. A, B, and C are three of the swimmers. What is the probability that A, B, and C will finish in the top three?

3. Three cards are drawn one after the other randomly from an ordinary deck of 52 cards without replacement. What is the probability that the first card is a black king, the second card is a red queen, and the third card is a black jack?

For solutions to **Activity 5**, turn to the **Appendix, Topic 1**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section. } You may decide to do both.

Extra Help

If you can do one thing in five different ways and another thing in seven different ways, you can do these two things in succession in $5 \times 7 = 35$ ways. This reasoning is called the fundamental counting principle. Some books call it the multiplicative counting principle.

Example 26

A committee has six members. In how many ways can they choose a chairperson and a secretary?

Solution:

Draw two boxes: one represents the number of ways to choose a chairperson, and the second one represents the number of ways to choose a secretary.
There are six different ways to choose a chairperson. Write 6 in the first box.
There are five persons left; therefore, there are five ways to choose a secretary. Write 5 in the second box.

6	5

There are $6 \times 5 = 30$ ways to choose a chairperson and a secretary.

Arranging things in a row is called linear arrangement or linear permutation. If you arrange five different objects in a row, the number of arrangements is $5 \times 4 \times 3 \times 2 \times 1$, and the product of the first five natural numbers can be denoted by the symbol 5! which is read as five factorial. If n different objects are arranged in a row, the number of arrangements is $n!$.

You have to know how to expand a factorial. Examine the next example.

Example 27

Expand $(n + 2)!$.

Solution:

$(n + 2)!$ is the product of the first $(n + 2)$ natural numbers.

$$(n+2)! = (n+2)(n+1)(n)(n-1)(n-2) \ldots 2 \times 1$$

Example 28

Expand $(n - r - 3)!$.

Solution:

$$(n-r-3)! = (n-r-3)(n-r-4)(n-r-5) \ldots 2 \times 1$$

If there are eight objects and you can choose any four of them to make arrangements, the number of arrangements is $8 \times 7 \times 6 \times 5$ and is denoted by ${}_8P_4$. This symbol can also be defined as follows:

$$\begin{aligned} {}_8P_4 &= \frac{8!}{(8-4)!} \\ &= \frac{8!}{4!} \\ &= 8 \times 7 \times 6 \times 5 \\ &= 1680 \end{aligned}$$

In general, ${}_nP_r = \frac{n!}{(n-r)!}$.

Some problems involve different kinds of permutations. The total number of permutations is the sum of the different permutations.

Example 29

How many different "words" can be formed using any number of letters in the word HOUR if no letter is repeated?

Solution:

If you use one letter to form a word, calculate ${}_4P_1$.

$$\begin{aligned} {}_4P_1 &= \frac{4!}{(4-1)!} \\ &= 4 \text{ ways} \end{aligned}$$

If you use two letters to form a word, calculate ${}_4P_2$.

$$\begin{aligned} {}_4P_2 &= \frac{4!}{(4-2)!} \\ &= 4 \times 3 \\ &= 12 \text{ ways} \end{aligned}$$

If you use three letters to form a word, calculate ${}_4P_3$.

$$\begin{aligned} {}_4P_3 &= \frac{4!}{(4-3)!} \\ &= 4 \times 3 \times 2 \times 1 \\ &= 24 \text{ ways} \end{aligned}$$

If you use four letters to form a word, calculate ${}_4P_4$.

$$\begin{aligned} {}_4P_4 &= \frac{4!}{(4-4)!} \\ &= 4! \\ &= 24 \text{ ways} \end{aligned}$$

The total number of permutations is 4 + 12 + 24 + 24 = 64.

Some questions involve permutations which affect each other, and the total number of permutations is the product of the permutations involved. Look at the following example.

Example 30

How many different licence plates can be made using three letters from the word PAIR followed by three digits from the number 234?

Solution:

The number of three-letter arrangements $= {}_4P_3$

$$= \frac{4!}{(4-3)!}$$

$$= 24$$

The number of three-digit arrangements $= {}_3P_3$

$$= 3!$$

$$= 6$$

By the fundamental counting principle, the total number of arrangements is $24 \times 6 = 144$.

Some arrangements involve more than one identical object. Look at the following example.

Example 31

How many distinguishable permutations can be made from all the letters of the word TEST?

Solution:

Since there are two T's, the arrangements of these two T's should be considered the same.

$$\text{The total number of arrangements} = \frac{4!}{2!} \quad \begin{matrix} \leftarrow \text{arrangements of 4 letters} \\ \leftarrow \text{arrangements of 2 T's} \end{matrix}$$

$$= 4 \times 3$$

$$= 12$$

In general, the number of permutations of n objects taken all at a time, of which u are alike, v are alike of another kind, etc., is as follows:

$$P = \frac{n!}{u!v!\ldots}, \text{ where } u+v+\ldots = n$$

If n objects are arranged in a circle, the number of arrangements is $(n-1)!$. For example, if you want to arrange eight different flags in a circle, the number of arrangements is $(8-1)! = 5040$ ways.

Arrangements like key chains or necklaces are called ring permutations. If you look at the arrangement from the opposite side (the back), it looks like a different arrangement, but you have never changed anything. Therefore, these two arrangements should be considered the same. The total number of arrangements is half the number of the original circular permutations.

$$\text{Ring permutations} = \frac{(n-1)!}{2}$$

Example 32

A key chain has five keys. In how many different ways can these keys be arranged?

Solution:

$$\text{The number of arrangements} = \frac{(5-1)!}{2}$$
$$= \frac{4!}{2}$$
$$= 12$$

Some probability questions involve permutations.

Recall that $\text{probability} = \frac{\text{number of favourable cases}}{\text{number of possible cases}}$.

Look at the next example.

Example 33

A bag contains all the four-digit numbers made with the five digits 1, 2, 3, 4, and 5 without repetition. A number is drawn randomly from the bag. What is the probability that the number is 2431?

Solution:

$$\text{Probability} = \frac{1}{{}_5P_4}$$
$$= \frac{1}{\frac{5!}{(5-4)!}}$$
$$= \frac{1}{5!}$$
$$= \frac{1}{120}$$

Do the following questions.

1. How many four-digit numbers can be made from the digits 2, 4, 6, 8, and 9 without repetition?

2. How many different "words" can be formed using any number of letters in the word MANY if no letter is repeated?

3. In how many ways can two boys and three girls be seated in a row if the two boys always occupy the first two chairs?

4. In how many ways can five red beads and three blue beads be arranged in a row?

5. In how many ways can ten different marbles be arranged in a circle?

6. A key chain has seven keys. In how many ways can these keys be arranged?

7. Two cards are drawn one after another from a standard deck of 52 cards. What is the probability that the first card is a club and the second card is a red king?

For solutions to **Extra Help**, turn to the **Appendix, Topic 1**.

Extensions

In this topic, you have learned a very important principle: the fundamental counting principle. It states that the number of ways of making two choices is $a \times b$ if the first choice can be made in a ways, and for each of these a ways, a second choice can be made in b ways. Since the result is the product of a and b, it is more appropriate to call it the multiplicative counting principle.
There is another basic principle called the additive counting principle. Both principles can be explained in terms of set.

Look at the multiplicative counting principle first.

If A and B are two sets of numbers such that $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$,
then $n(A)$ is the number of elements in A, and $n(B)$ is the number of elements in B. In this case, $n(A) = 3$ and $n(B) = 4$.

The Cartesian product $A \times B = \{(a,b) \mid a \in A,\ b \in B\}$ is the set of all the ordered pairs (a, b), and the number of elements in $A \times B$ is $n(A \times B) = n(A) \times n(B)$.

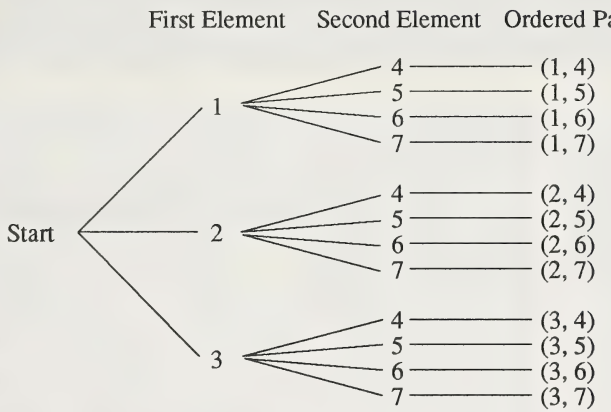
In this case, $n(A \times B) = 3 \times 4$

$= 12.$

Therefore, there are twelve ordered pairs in the set of $A \times B$.

$\in$ is a letter of the Greek alphabet called epsilon which means belongs to.

A tree diagram is used to show this Cartesian product.



Now look at the additive counting principle.

If A and B are two sets such that set A has $n(A)$ elements and set B has $n(B)$ elements, then

- $n(A \cup B) = n(A) + n(B)$, if $A \cap B = \phi$
- $n(A \cup B) = n(A) + n(B) - n(A \cap B)$, if $A \cap B \neq \phi$

Consider the two sets $A = \{1, 2, 3\}$ and $B = \{4, 5, 6, 7\}$.

The intersection of A and $B = A \cap B = \phi$ because these two sets do not have any elements in common.

$n(A) = 3,\ n(B) = 4$

$\therefore n(A \cup B) = 3 + 4$

$= 7$

The total number of different elements in the union of these two sets is 7.

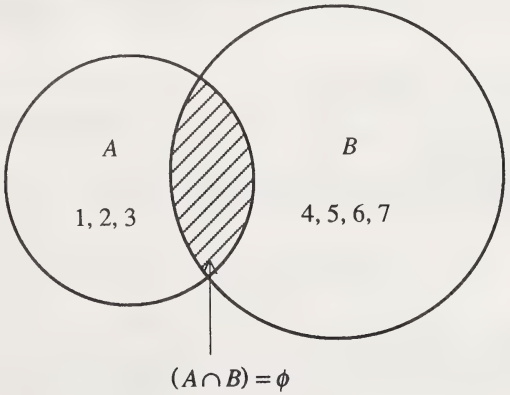
$A \cup B$ = the union of A and B

$A \cap B$ = the intersection of A and B

ϕ = empty set, a set which has no element

$A \cup B$ $A \cap B$

Use a diagram to illustrate this situation. Let a circle A represent set A and another circle B represent set B.

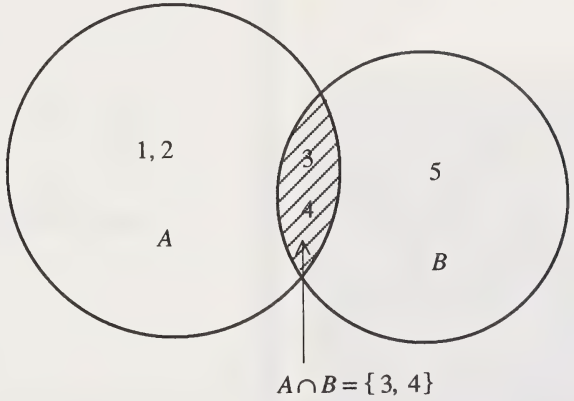


Consider the two sets $A = \{1, 2, 3, 4\}$ and $B = \{3, 4, 5\}$.
The intersection of A and $B = A \cap B = \{3, 4\}$ which is not an empty set.

$$n(A) = 4, \quad n(B) = 3, \quad n(A \cap B) = 2$$
$$\therefore n(A \cup B) = n(A) + n(B) - n(A \cap B)$$
$$= 4 + 3 - 2$$
$$= 5$$

The total number of different elements in the union of these two sets is 5.

The following diagram illustrates this situation.



Do the following questions.

1. If $A = \{m, n, o\}$ and $B = \{h, f, g\}$, draw a tree diagram to illustrate the Cartesian product $A \times B$.

2. If $M = \{2, 4, 6, 8, 10\}$, $P = \{1, 3, 5\}$, and $Q = \{1, 2, 3, 4\}$, find $n(P \cap Q)$, $n(M \cap P)$, $n(M \cup P)$, and $n(P \cup Q)$. Draw a diagram to illustrate each of them.

For solutions to **Extensions**, turn to the **Appendix, Topic 1**.

Topic 2 Combinations

Introduction

Permutation means arrangement. Order is very important in permutation.

If order is not important, only selection of different objects is important. This kind of selection is called combination. Combination is used to solve probability problems and is also used to explain binomial expansion. You are going to find all of these in this topic.

What Lies Ahead

Throughout this topic you will learn to

1. define the meaning and use applications of the formula

   $${}_nC_r = \frac{n!}{(n-r)!r!}$$

2. solve probability questions which involve the use of combinations

3. define and use the binomial theorem and Pascal's triangle

Now that you know what to expect, turn the page to begin your study of combinations.

Exploring Topic 2

Activity 1

Define the meaning and use applications of the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$.

If a student council wants to select a three-member committee to organize a graduation party, it is important to know who is selected, but the order of the students' selection is not relevant. Selections of this kind are called **combinations**.

Consider the four letters *a*, *b*, *c*, and *d*. In how many different ways can three letters be chosen from *a*, *b*, *c*, and *d* ? The possible solutions are as follows:

(a, b, c) (b, c, d) (c, d, a) (d, a, b)

As you can see, there are four different ways to do it. If you try to arrange the letters in each group, each group can have 3! arrangements. The total number of arrangements should be 4 groups × 3! = 24 and it should be the same as ${}_4P_3$. These four groups represent the number of combinations. If you use a new symbol ${}_nC_r$ to represent the number of combinations of *n* objects taken *r* at a time, then ${}_4C_3 = 4$.

The following is also true:

$$\begin{aligned} {}_4C_3 \times 3! &= 24 \\ &= {}_4P_3 \\ &= \frac{4!}{(4-3)!} \\ \therefore {}_4C_3 &= \frac{4!}{(4-3)!3!} \end{aligned}$$

In general, for all natural numbers *n* and *r*, $n \geq r$.

$${}_nC_r \times r! = {}_nP_r$$

$${}_nC_r \times r! = \frac{n!}{(n-r)!}$$

Hence, $\boldsymbol{{}_nC_r = \frac{n!}{(n-r)!r!}}$.

${}_nC_r$ can also be written as $\binom{n}{r}$.

There is a problem. What is the number of combinations of *n* things taken zero at a time?

$$\begin{aligned} {}_nC_0 &= \frac{n!}{(n-0)!0!} \\ &= \frac{n!}{n!} \\ &= 1 \end{aligned}$$

Do you agree that there is only one way to choose nothing? Note that 0! = 1.

Look at the following examples.

Example 1

Evaluate ${}_7C_5$.

Solution:

$$
\begin{aligned}
{}_7C_5 &= \frac{7!}{(7-5)!5!} \\
&= \frac{7!}{2!5!} \\
&= \frac{7\times 6}{2!} \\
&= 21
\end{aligned}
$$

Example 2

Evaluate ${}_5C_1 + {}_3C_2 \times {}_4C_1 - {}_6C_0$.

Solution:

$$
\begin{aligned}
&{}_5C_1 + {}_3C_2 \times {}_4C_1 - {}_6C_0 \\
&= \frac{5!}{(5-1)!1!} + \frac{3!}{(3-2)!2!} \times \frac{4!}{(4-1)!1!} - \frac{6!}{(6-0)!0!} \\
&= \frac{5}{1} + \frac{3}{1} \times \frac{4}{1} - 1 \\
&= 16
\end{aligned}
$$

Example 3

A committee of four students is to be chosen from a group of ten students. In how many ways can this be done?

Solution:

There are ten students; thus, $n = 10$.
Four students are to be chosen; therefore, $r = 4$.

$$
\begin{aligned}
{}_{10}C_4 &= \frac{10!}{(10-4)!4!} \\
&= \frac{10!}{6!4!} \\
&= \frac{10\times 9\times 8\times 7}{4\times 3\times 2\times 1} \\
&= 210
\end{aligned}
$$

There are 210 ways to form a committee.

The next example involves mutually exclusive cases. The occurrence of one of them excludes the occurrence of the others. The word **or** will be used in the question. When you say that you want to select one kind **or** another, the selection of one kind of object excludes the selection of the other kind. The total number of selections is the sum of all the different selections.

Example 4

There are seven books on the bookshelf. In how many ways can five **or** more books be selected?

Solution:

The statement five or more means you can select five books, six books, or seven books.

If you select five books, the number of selections is ${}_7C_5$.

If you select six books, the number of selections is ${}_7C_6$.

If you select seven books, the number of selections is ${}_7C_7$.

Altogether there are ${}_7C_5+{}_7C_6+{}_7C_7$ ways to choose the books.

$$
\begin{aligned}
{}_7C_5+{}_7C_6+{}_7C_7 &= \frac{7!}{2!5!}+\frac{7!}{1!6!}+\frac{7!}{0!7!} \\
&= \frac{7\times 6}{1\times 2}+7+1 \\
&= 29 \text{ ways}
\end{aligned}
$$

The next example is somewhat tricky.

Example 5

In how many ways can six different books be separated into two groups, each of three books?

Solution:

Three books can be selected from the six books in ${}_6C_3$ ways. However, the books are separated into two groups of three books each. In other words, every time you select a group of three books, you actually get two groups of three books each. Therefore, the second group should not be counted again.

$$
\begin{aligned}
\text{Thus, the number of ways} &= \frac{{}_6C_3}{2} \\
&= \frac{\frac{6!}{3!3!}}{2} \\
&= \frac{6\times 5\times 4}{3\times 2\times 2} \\
&= 10 \text{ ways}
\end{aligned}
$$

The next example is a problem involving multiple selection from different sources. The word **and** will be used in the question.

Whenever a combination problem involves dividing a group of objects into two equal parts, you must divide ${}_nC_r$ by 2 in order to avoid counting identical cases twice.

Example 6

A person goes to a department store to buy two shirts, two pairs of dress pants, and three pairs of socks. The store has eight shirts, ten pairs of dress pants, and fifteen pairs of socks, all of different styles or colours. How many different selections of shirt, dress pants, and socks can this person purchase?

Solution:

There are ${}_8C_2$ selections of shirts, ${}_{10}C_2$ selections of dress pants, and ${}_{15}C_3$ selections of socks. For every selection of shirt, there are ${}_{10}C_2$ selections of dress pants, and for every selection of shirt and dress pants, there are ${}_{15}C_3$ selections of socks.

By the fundamental counting principle, you have to multiply all the selections in order to determine the total number of combinations.

$$\begin{aligned} {}_8C_2 \times {}_{10}C_2 \times {}_{15}C_3 &= \frac{8!}{(8-2)!2!} \times \frac{10!}{(10-2)!2!} \times \frac{15!}{(15-3)!3!} \\ &= \frac{8!}{6!2!} \times \frac{10!}{8!2!} \times \frac{15!}{12!3!} \\ &= 28 \times 45 \times 455 \\ &= 573\,300 \text{ combinations} \end{aligned}$$

If a problem involves multiple selections from the same source, you also multiply the selections. Look at the following example.

Example 7

A bag of twenty different marbles is to be divided into a group of ten marbles, a group of six marbles, and a group of four marbles. In how many ways can this be done?

Solution:

The first selection affects the second and third selections. The second selection also affects the third selection.

There are twenty different marbles. The first group can be selected in ${}_{20}C_{10}$ ways. Now, only ten marbles are left. Six marbles are chosen from the remaining ten marbles. The second group can be selected in ${}_{10}C_6$ ways. The third group can be chosen from the remaining four marbles in ${}_4C_4$ ways. Again, you multiply the selections.

$$\begin{aligned} {}_{20}C_{10} \times {}_{10}C_6 \times {}_4C_4 &= \frac{20!}{10!10!} \times \frac{10!}{6!4!} \times \frac{4!}{4!0!} \\ &= 38\,798\,760 \text{ ways} \end{aligned}$$

If any number of objects are selected from n objects, each object is either selected or excluded. For example, four objects a, b, c, and d are available. You want to know the number of ways to choose 1, 2, 3, 4, or no objects. Since each object can be either selected or excluded, by the fundamental counting principle, the total number of selections is the product of $2 \times 2 \times 2 \times 2 = 2^4$. This includes the case in which no object is chosen.

In general, if there are n objects, the total number of ways to select any number of objects is 2^n, which includes the case in which no object is chosen.

If you do not want this case in which no object is chosen to be included, then subtract 1 from 2^n. The number of ways to select any number of objects except zero is $2^n - 1$.

Example 8

In how many ways can a boy choose one or more of five different toys?

Solution:

Each toy can be dealt with in two ways. It can be chosen or rejected. Since the case in which no toy is chosen is not considered as a possible outcome, the number of ways is $2^5 - 1 = 32 - 1 = 31$ ways.

Some problems involve different kinds of restricted selections of objects. Study the next four examples carefully.

Example 9

A committee of four students is to be chosen from a group of twelve students including the president and secretary of the students' union.

- In how many ways can this be done if the president is to be included in every selection?

Solution:

Since the president is included in every selection, three more students must be selected from the remaining eleven students.
The total number of selections $= {}_1C_1 \times {}_{11}C_3$

$$= 1 \times \frac{11!}{8!3!}$$
$$= \frac{11 \times 10 \times 9}{3 \times 2 \times 1}$$
$$= 165 \text{ ways}$$

- In how many ways can this be done if the president and the secretary must be included?

Solution:

Since the president and the secretary must be included, two more students must be selected from the remaining ten students.
The total number of selections $= {}_2C_2 \times {}_{10}C_2$

$$= 1 \times \frac{10!}{8!2!}$$
$$= 45 \text{ ways}$$

- In how many ways can this be done if the president and the secretary must be excluded and one other student must be included?

Solution:

Since one is always included and two are always excluded, three more students must be chosen from the remaining nine students.

The total number of selections $= {}_1C_1 \times {}_9C_3$

$$= \frac{9!}{6!3!}$$

$$= \frac{9 \times 8 \times 7}{3 \times 2 \times 1}$$

$$= 84 \text{ ways}$$

Example 10

A class has twenty students: nine girls and eleven boys. A committee of three students is to be selected so as to include at least one girl. In how many ways can this be done?

Solution:

Instead of finding all of the possible combinations, it is easier to use a subtraction method. Subtract the number of selections in which no girl is included from the number of selections of any three students with no restrictions.

The number of ways in which any three students can be selected out of twenty students is ${}_{20}C_3$.

The number of ways in which three students can be selected so that no girl is included is ${}_{20-9}C_3 = {}_{11}C_3$.

The number of ways in which three students can be selected so that at least one girl is included $= {}_{20}C_3 - {}_{11}C_3$

$$= \frac{20!}{17!3!} - \frac{11!}{8!3!}$$

$$= \frac{20 \times 19 \times 18}{3 \times 2 \times 1} - \frac{11 \times 10 \times 9}{3 \times 2 \times 1}$$

$$= 1140 - 165$$

$$= 975 \text{ ways}$$

Example 11

There are eight coplanar points. Three of them are collinear. No three of the other five points are in a straight line. How many straight lines can be formed by joining any two of the eight points?

Solution:

If no three of the eight points were collinear, the number of lines formed would be ${}_8C_2$.

Other method:

$${}_9C_1 \times {}_{11}C_2 + {}_9C_2 \times {}_{11}C_1 + {}_9C_3$$

↑ one girl ↑ two girls ↑ three girls

$$= 495 + 396 + 84$$

$$= 975$$

Coplanar means in the same plane.

Use the subtraction method when **at least** is used in the question.

The number of lines formed by three noncollinear points is ${}_3C_2$.

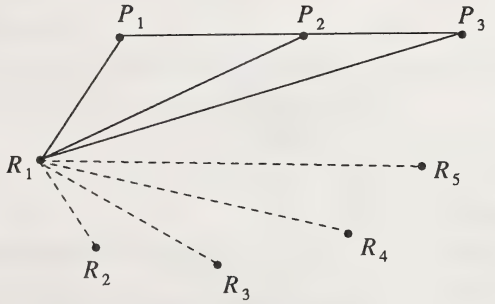
Since the three points are collinear, they form one straight line.

The number of straight lines formed by joining the eight points if three of them are collinear $= {}_8C_2 - {}_3C_2 + 1$

$$= \frac{8!}{6!2!} - \frac{3!}{1!2!} + 1$$
$$= 28 - 3 + 1$$
$$= 26$$

Alternate Solution:

In the diagram below, P_1, P_2, and P_3 represent the three collinear points. The noncollinear points are represented by R_1, R_2, R_3, R_4, and R_5.



If R_1 is joined to each collinear point (P_1, P_2, P_3), three lines are formed. When each of the five noncollinear points $(R_1, R_2, R_3, R_4, R_5)$ are joined to each of the three collinear points (P_1, P_2, P_3), the number of lines formed is $5 \times 3 = 15$.

One line is produced by joining any two of the noncollinear points $(R_1, R_2, R_3, R_4, R_5)$. The total number of lines produced by the five noncollinear points is ${}_5C_2 = \frac{5!}{3!2!} = 10$.

The three collinear points P_1, P_2, and P_3 form one straight line.

The total number of lines formed $= 15 + 10 + 1$
$= 26$

Example 12

Determine the number of diagonals in a nonagon (nine sides).

Solution:

A nonagon has nine vertices. The number of lines formed by joining any two of these vertices including the sides is ${}_9C_2$.

Number of diagonals = number of lines – number of sides

$$= {}_9C_2 - 9$$
$$= \frac{9!}{7!2!} - 9$$
$$= 36 - 9$$
$$= 27$$

Now do at least the first thirteen questions.

1. Evaluate ${}_{17}C_3$.

2. Evaluate ${}_4C_3 + {}_5C_0 - {}_6C_1$.

3. Show that ${}_6C_3 + {}_6C_4 = {}_7C_4$.

4. There are ten different pictures. In how many ways can seven or more pictures be selected?

5. In how many ways can ten basketball players be separated into two groups, each of five players?

6. A person goes to a grocery store to buy two kinds of cookies, one carton of milk, three kinds of vegetables, and four different kinds of soups. If the store has three different kinds of milk, eight different kinds of cookies, ten different kinds of vegetables, and nine different kinds of soups, in how many different ways can this person buy the groceries?

7. In how many ways can a boy choose no more than three of twelve different cookies? (He must choose at least one cookie.)

8. A class of twenty-one students is to be divided into a group of nine students, a group of seven students, and a group of five students. In how many ways can this be done?

9. In how many ways can a little girl choose one or more of ten different candies?

10. A political party has eight candidates. In how many possible ways can 0, 1, 2, . . . , or 8 candidates be elected?

11. There are twelve face cards (four kings, four queens, and four jacks) in a standard deck of 52 cards. How many five-card hands can be formed under the following conditions?

    a. No face card is to be included.
    b. Two and only two face cards must be included.
    c. The kings are always excluded and the two red queens are always included.
    d. At least one king is included. (See Example 10.)

12. There are twelve coplanar points. Four of them are collinear. No three of the other eight points are in a straight line. How many straight lines can be formed by joining any two of the twelve points?

13. Determine the number of diagonals in a fourteen-sided polygon.

14. In how many different ways can three or more students be chosen out of fifteen students?

15. Tui has five shirts, six pairs of dress pants, and three hats. How many selections of shirts, dress pants, and hats can be made, taking at least one shirt and one pair of dress pants?

16. A hockey league consists of nine teams. Each team plays exactly one game with each of the other teams. How many league games will be played?

For solutions to **Activity 1**, turn to the **Appendix, Topic 2**.

Activity 2

Solve probability questions which involve the use of combinations.

In probability, some sample spaces involve a large number of elements. It is impractical to list all the elements of a large sample space. Permutations and combinations can be used to find the number of favourable or possible cases. Look at the following examples.

Example 13

Three cards are drawn at random from a well-shuffled ordinary deck of 52 cards. Find the probability that all three cards are kings.

Solution:

The number of ways in which three cards can be selected out of 52 cards is
${}_{52}C_3 = \frac{52!}{49!3!} = 22\,100 = n(S)$.

The number of ways in which three kings can be selected out of four kings is ${}_4C_3 = \frac{4!}{3!1!} = 4 = n(A)$.

$\therefore$ The required probability is $P(E) = \frac{4}{22\,100}$

$= \frac{1}{5525}$.

You can solve the same problem using permutations.

The number of arrangements of 52 cards, taken three at a time $= {}_{52}P_3$

$= \frac{52!}{(52-3)!}$

$= 132\,600$ ways

The number of arrangements of four kings, taken three at a time $= {}_4P_3$

$= \frac{4!}{1!}$

$= 24$ ways

The required probability is $P(E) = \frac{24}{132\,600}$

$= \frac{1}{5525}$.

Example 14

There are twelve boys and eight girls in a class. If a group of five students is chosen at random to play a game, find the probability that two of them are girls.

Solution:

The number of ways in which five students can be chosen out of twenty $= {}_{20}C_5$

$$= \frac{20\times19\times18\times17\times16}{5\times4\times3\times2\times1}$$
$$= 15\,504$$

The number of ways in which two girls and three boys can be chosen $= {}_{12}C_3 \times {}_8C_2$

$$= \frac{12!}{9!3!} \times \frac{8!}{6!2!}$$
$$= 220\times28$$
$$= 6160$$

The required probability is $P(E) = \frac{6160}{15\,504}$

$$= \frac{385}{969}.$$

Some questions involve more than one event. When the word **or** is used in the question, you have to apply the addition theorem.

Example 15

If two cards are drawn at random from an ordinary deck of 52 cards, what is the probability that both are kings or both are jacks?

Solution:

The number of combinations of 52 cards taken two at a time is

$${}_{52}C_2 = \frac{52!}{50!2!}$$
$$= 1326.$$

The number of combinations of four kings taken two at a time is

$${}_4C_2 = \frac{4!}{2!2!}$$
$$= 6.$$

The number of combinations of four jacks taken two at a time is

$${}_4C_2 = \frac{4!}{2!2!}$$
$$= 6.$$

The required probability $= \frac{6}{1326} + \frac{6}{1326}$

$$= \frac{12}{1326}$$
$$= \frac{2}{221}$$

Complementary Events

The symbol $\overline{E}$ means the complement of E.
If a sample space is separated into two parts, E and $\overline{E}$, then E is the complement of $\overline{E}$, and $\overline{E}$ is also the complement of E. The probability of a complementary event $P(\overline{E})$ can be determined by subtracting the probability of event E from one: $P(\overline{E}) = 1 - P(E)$.

The following example involves complementary events.

Example 16

Three toy cars are chosen at random from nine toy cars of which four are defective. Find the probability that at least one is defective.

Solution:

In order to find the probability that at least one toy car is defective, you have to find the probability that no toy car is defective. Subtract this probability from 1, and this gives the probability that at least one toy car is defective.

If four toy cars are defective, there are five cars which are not defective.
The number of ways to choose three nondefective toy cars is

$$_5C_3 = \frac{5!}{2!3!}$$
$$= 10 \text{ ways.}$$

The number of ways to choose any three toy cars is

$$_9C_3 = \frac{9!}{6!3!}$$
$$= 84 \text{ ways.}$$

The probability that no toy car is defective is $\frac{10}{84}$.

The probability that at least one toy car is defective $= 1 - \frac{10}{84}$

$$= \frac{74}{84}$$
$$= \frac{37}{42}$$

Do the following questions.

1. A bag contains nine cards, numbered from 1 to 9 inclusive. Two cards are drawn at random. What is the probability that both cards are even?

2. A person bought two of a total of fifty raffle tickets. If there are two winning tickets, what is the probability that this person has exactly one winning ticket?

3. A coin is tossed; then, a die is rolled. What is the probability that a tail and an odd number turn up?

4. A bag contains twelve cards, numbered from 1 to 12 inclusive. Two cards are drawn at random. What is the probability that both are even **or** both are greater than 5?

5. There are fifteen boys and ten girls in a class. A yearbook committee of four is chosen at random. What is the probability that at least one girl is chosen?

For solutions to **Activity 2**, turn to the **Appendix, Topic 2**.

Activity 3

Define and use the binomial theorem and Pascal's triangle.

A binomial is a polynomial which has two terms. For example, $(a + b)$ is a binomial of degree 1, and $(a+b)^2$ is a binomial of degree 2. In this section, you are going to investigate the expansion of binomials of the form $(a+b)^2$, $(a+b)^3$, and so on. Consider the following expansions:

$$(a+b)^2=(a+b)(a+b)=a^2+ab+ba+b^2=a^2+2ab+b^2$$

Each term of the expansion is obtained by multiplying a letter (a or b) from each of the two factors $(a + b)$. The first term of the product $\left(a^2\right)$ has no b's. The second term $(2ab)$ has one b, and the third term $\left(b^2\right)$ has two b's.

$$\begin{aligned}(a+b)^3&=(a+b)(a+b)(a+b)\\&=aaa+aab+aba+abb+baa+bab+bba+bbb\\&=a^3+3a^2b+3ab^2+b^3\end{aligned}$$

Again, each term of the expansion is obtained by multiplying a letter (a or b) from each of the three factors $(a + b)$. The first term of the product $\left(a^3\right)$ has no b's. The second term $\left(3a^2b\right)$ has one b. The third term $\left(3ab^2\right)$ has two b's. The fourth term $\left(b^3\right)$ has three b's.

Each term looks like a combination of a's and b's as if all the a's and b's were different.

The first term has no b's. It is like choosing no b's from three b's. The combination ${}_3C_0=1$ is the coefficient of the first term a^3.

The second term has one b. It is like choosing one b from three b's. The combination ${}_3C_1=3$ is the coefficient of the second term $3a^2b$.

The third term has two b's. It is like choosing two b's from three b's. The combination ${}_3C_2=3$ is the coefficient of the third term $3ab^2$.

The fourth term has three b's. It is like choosing three b's from three b's. The combination ${}_3C_3=1$ is the coefficient of the fourth term b^3.

If you use the same reasoning to expand $(a+b)^4$, then

$$\begin{aligned}(a+b)^4&={}_4C_0a^4+{}_4C_1a^3b+{}_4C_2a^2b^2+{}_4C_3ab^3+{}_4C_4b^4\\&=a^4+4a^3b+6a^2b^2+4ab^3+b^4\end{aligned}$$

(If you wish, multiply $(a + b)(a + b)(a + b)(a + b)$. The answer should be the same.)

Using the same reasoning, you can expand $(a+b)^n$.

The first term has no b's. The coefficient is ${}_nC_0=1$ and a^n will be the first term.

The second term has one b. The coefficient is ${}_nC_1=n$ and $na^{n-1}b$ will be the second term.

The third term has two b's. The coefficient is ${}_nC_2$ and ${}_nC_2a^{n-2}b^2$ will be the third term.

Continued this way, the rth term has $(r-1)$ b's. The coefficient is ${}_nC_{r-1}$ and ${}_nC_{r-1}\,a^{n-r+1}\,b^{r-1}$ **is the rth term.**

The last term has n b's. The coefficient of the last term is ${}_nC_n$ and ${}_nC_n\,b^n$ is the nth term.

Therefore,

$$(a+b)^n = {}_nC_0\,a^n + {}_nC_1\,a^{n-1}b + {}_nC_2\,a^{n-2}b^2 + \ldots + {}_nC_{r-1}\,a^{n-r+1}\,b^{r-1} + \ldots + {}_nC_n\,b^n.$$

This expansion is called the **binomial theorem.**

Look at the following examples.

Example 17

Use the binomial theorem to expand $(a+b)^6$.

Solution:

$$(a+b)^6 = {}_6C_0\,a^6 + {}_6C_1\,a^5b + {}_6C_2\,a^4b^2 + {}_6C_3\,a^3b^3 + {}_6C_4\,a^2b^4 + {}_6C_5\,ab^5 + {}_6C_6\,b^6$$
$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6a^1b^5 + b^6$$

Example 18

Determine the first three terms in the expansion of $(3x-5)^{10}$ by using the binomial theorem.

Solution: $(3x-5)^{10} = \left[(3x)+(-5)\right]^{10}$

$$= {}_{10}C_0(3x)^{10} + {}_{10}C_1(3x)^9(-5) + {}_{10}C_2(3x)^8(-5)^2 + \ldots$$
$$= 59\,049x^{10} + (10)(19\,683)x^9(-5) + (45)(6561)x^8(25) + \ldots$$
$$= 59\,049x^{10} - 984\,150x^9 + 7\,381\,125x^8 \ldots$$

Example 19

Determine the seventh term (t_7) in the expansion of $(2x-3)^{11}$.

Solution:

The rth term in the expansion of $(a+b)^n$ is ${}_nC_{r-1}a^{n-r+1}b^{r-1}$.

$n=11,\ r=7,\ a=2x,\ b=(-3)$

$$\begin{aligned} t_7 &= {}_{11}C_{7-1}(2x)^{11-7+1}(-3)^{7-1} \\ &= {}_{11}C_6(2x)^5(-3)^6 \\ &= \frac{11!}{5!6!}(2x)^5(-3)^6 \\ &= 462(32x^5)(729) \\ &= 10\ 777\ 536x^5 \end{aligned}$$

Example 20

What is the numerical coefficient, in the form of ${}_nC_r$, for the term x^5y^7 in the expansion of $(x+y)^{12}$?

Solution:

x^5y^7 is the eighth term in the expansion $(7+1=8)$.
Thus, the coefficient of the eighth term is ${}_{12}C_7$.

Example 21

${}_{15}C_{r-1}x^3y^{r-1}$ is a term in the expansion of $(x+y)^{15}$. What is the value of r?

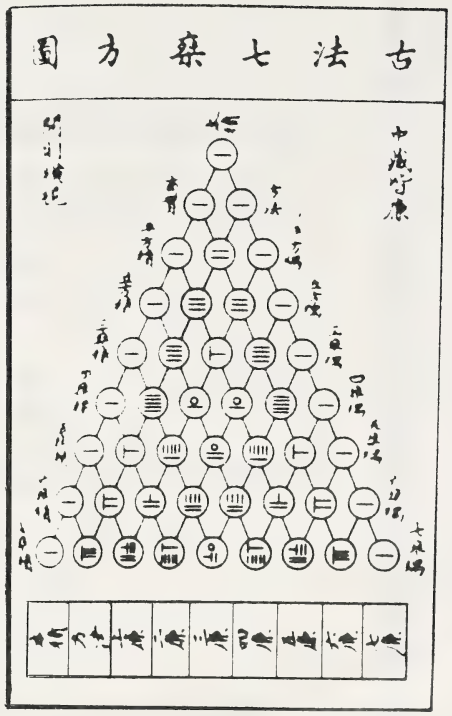
Solution:

In the expansion $(x+y)^n$, the sum of the exponents for any one term in the expansion is always equal to the value n.

$$\begin{aligned} 3+r-1 &= 15 \\ \therefore r &= 15-3+1 \\ &= 13 \end{aligned}$$

The numerical coefficients in the binomial expansion $(a+b)^n$ can be written in the form of a triangular array. This array is called **Pascal's triangle**. Blaise Pascal developed this triangle in the seventeenth century. Actually, the Chinese developed such an array in the tenth century. It was mentioned in a twelfth century Chinese manuscript (see margin box). However, Pascal was the one who developed further applications of the triangle.

Blaise Pascal



If you arrange the numerical coefficients in the binomial expansion $(a+b)^n$ in a triangular array, you can see some interesting patterns.

$(a+b)^0 \quad n=0$

$(a+b)^1 \quad n=1$

$(a+b)^2 \quad n=2$

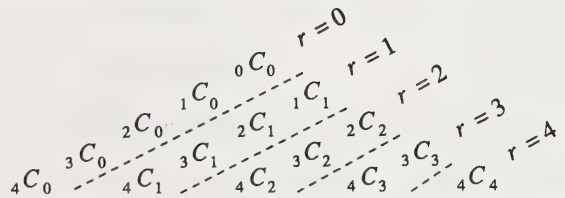
$(a+b)^3 \quad n=3$

$(a+b)^4 \quad n=4$

. .

. .

. .



If you evaluate each $_nC_r$, the triangle becomes as follows:

$n=0$ 1

$n=1$ 1 1

$n=2$ 1 2 1

$n=3$ 1 3 3 1

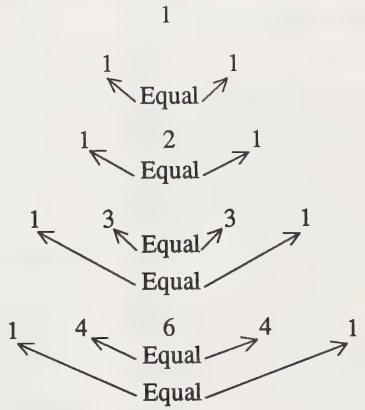
$n=4$ 1 4 6 4 1

. .

. .

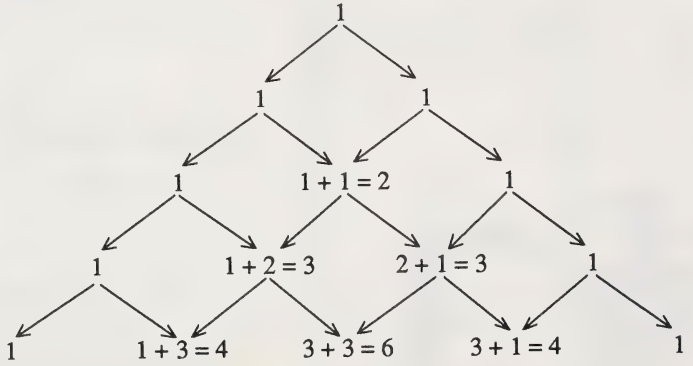
. .

In any row of this triangle, the numbers which are equidistant from the centre are equal.



From this pattern, you can see that $_nC_r = {}_nC_{n-r}$. For example, when $n=4$ and $r=1$, $_4C_1 = {}_4C_3$ since $4=4$.

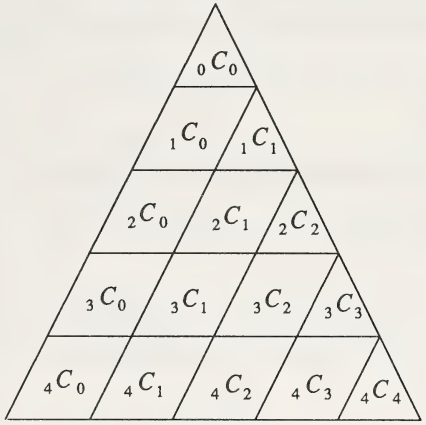
You can extend the triangle by adding two adjacent numbers to obtain the number in the row below. For example,



It is quite easy to extend Pascal's triangle this way. This pattern shows that the sum of two adjacent numbers is a number in the row below.
In general, ${}_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}$.

This is called Pascal's theorem.

From Pascal's triangle shown below, ${}_3C_1$ and ${}_3C_2$ will be used to illustrate Pascal's theorem which states that the sum of two adjacent numbers in a horizontal row is equal to the number centred between and below the two adjacent numbers. This number is located in the horizontal row immediately below the two adjacent numbers.



$${}_3C_1 + {}_3C_2 = \frac{3!}{2!1!} + \frac{3!}{1!2!}$$
$$= 3 + 3$$
$$= 6$$

But, ${}_4C_2 = \frac{4!}{2!2!} = 6$.

Thus, ${}_4C_2 = {}_3C_1 + {}_3C_2$.

The general form of Pascal's theorem can be stated as

${}_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}$.

Study the following examples.

Example 22

What is the equivalent expression of ${}_{15}C_8$ in the form ${}_nC_r$?

Solution:

$${}_nC_r = {}_nC_{n-r}$$
$$\therefore {}_{15}C_8 = {}_{15}C_7$$

Example 23

What is the equivalent expression of ${}_{13}C_7 + {}_{13}C_8$ in the form ${}_nC_r$?

Solution:

$${}_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}$$
$$\therefore {}_{14}C_8 = {}_{13}C_7 + {}_{13}C_8$$

Do at least the first eight questions.

1. Use the binomial theorem to expand each of the following:

   a. $(x+2y)^8$

   b. $(2x-y)^7$

   c. $\left(x^2-\frac{1}{2}y\right)^5$

2. Determine the ninth term in the expansion of $(x-2)^{15}$.

3. Determine the middle term of $\left(x^2-\sqrt{y}\right)^6$.

4. What is the numerical coefficient, in the form of ${}_nC_r$, for the term a^8b^8 in the expansion of $(a-b)^{16}$?

5. If the numerical coefficient is not stated, $a^x b^7$ is a term in the expansion of $(a+b)^{12}$. What is the value of x?

6. Which term in the expansion of $\left(\frac{1}{2}x-y\right)^6$ is $\frac{15}{16}x^4y^2$?

7. What is the equivalent expression of ${}_{14}C_6$ in the form ${}_nC_r$?

8. Write another expression in the form ${}_nC_r$ which is equal to ${}_{17}C_9+{}_{17}C_{10}$.

9. Use the binomial theorem to expand $(a+b-c)^4$.

10. Use the binomial expansion of $(x+y)^2$ to prove that
    $2^7={}_7C_0+{}_7C_1+{}_7C_2+{}_7C_3+{}_7C_4+{}_7C_5+{}_7C_6+{}_7C_7$.

For solutions to **Activity 3**, turn to the **Appendix, Topic 2**.

If you require help, do the Extra Help section.

If you want more challenging explorations, do the Extensions section.

} You may decide to do both.

Extra Help

When r objects are selected from n objects $(n \geq r)$ and the order of the objects is not important, this selection is called a combination. The symbol used to denote the number of combinations of n objects taken r at a time is ${}_nC_r$ and ${}_nC_r = \frac{n!}{(n-r)!r!}$.

Example 24

In how many ways can five colouring pencils be chosen from twelve different colouring pencils?

Solution:

The order in which the colouring pencils are chosen is not important.

The number of combinations $= {}_{12}C_5$

$$= \frac{12!}{(12-5)!5!}$$

$$= \frac{12!}{7!5!}$$

$$= \frac{12 \times 11 \times 10 \times 9 \times 8 \times \cancel{7!}}{\cancel{7!}5 \times 4 \times 3 \times 2 \times 1}$$

$$= 792 \text{ ways}$$

If a question involves more than one kind of selection and no two kinds of selections can occur at the same time, the word **or** will be used in the question, and the total number of selections is the sum of the numbers of different selections.

Example 25

There are 24 colouring pencils in a box. In how many ways can 20 or more pencils be selected?

Solution:

You may select 20, 21, 22, 23, or 24 pencils.

The total number of selections

$$= {}_{24}C_{20} + {}_{24}C_{21} + {}_{24}C_{22} + {}_{24}C_{23} + {}_{24}C_{24}$$

$$= \frac{24!}{4!20!} + \frac{24!}{21!3!} + \frac{24!}{22!2!} + \frac{24!}{23!1!} + \frac{24!}{24!0!} \qquad \text{(Note: } 0! = 1.\text{)}$$

$$= \frac{24 \times 23 \times 22 \times 21}{4 \times 3 \times 2 \times 1} + \frac{24 \times 23 \times 22}{3 \times 2 \times 1} + \frac{24 \times 23}{2 \times 1} + \frac{24}{1} + 1$$

$$= 10\,626 + 2024 + 276 + 24 + 1$$

$$= 12\,951$$

When different selections are made from different sources, the number of combinations is the product of the combinations. The word **and** will usually be used in the question.

Example 26

A boy has six different colouring pencils and four sheets of different coloured paper. He needs one colouring pencil and one sheet of paper. In how many ways can he select the pencil and paper?

Solution:

To select a colouring pencil, there are ${}_6C_1$ ways.

To select a sheet of paper, there are ${}_4C_1$ ways.

The total number of selections is ${}_6C_1 \times {}_4C_1 = 24$ ways.

The total number of ways to select any number of n objects is 2^n which includes the case in which no object is selected. If you want the number of ways to select one or more of n objects, the total number is $2^n - 1$.

Example 27

In how many ways can a student choose one or more of six courses?

Solution:

Each course can be chosen or rejected.
The student wants to choose at least one course.
The total number of ways is $2^6 - 1 = 63$ ways.

When **at least** is used in the question, usually it means that many cases are involved. It would be easier to use the subtraction method as shown in the example below.

Example 28

There are ten mathematics books and eight chemistry books. In how many ways can five books be chosen with at least one mathematics book?

Solution:

There are eighteen books altogether.
The number of ways to choose any five books is ${}_{18}C_5$.
The number of ways to choose no mathematics books equals the number of ways to choose five chemistry books which is ${}_8C_5$.

The number of ways to choose five books with at least one mathematics book $= {}_{18}C_5 - {}_8C_5$

$= 8568 - 56$

$= 8512$

Some probability questions involve the use of combinations.

Example 29

Eleven cards are numbered from 1 to 11 and put into a box. Two cards are drawn at random. What is the probability that both cards are odd numbers?

Solution:

The number of combinations of eleven cards taken two at a time is ${}_{11}C_2$. The six odd numbers are 1, 3, 5, 7, 9, and 11.

The number of combinations in which two odd-number cards can be chosen out of six is ${}_6C_2$.

The required probability is $P(E)=\frac{{}_6C_2}{{}_{11}C_2}=\frac{15}{55}=\frac{3}{11}$.

There are other probability questions which involve the use of combinations. Please review the other examples in Activity 2 of this topic.

Combination is also used to calculate the coefficients of a binomial expansion.

$(a+b)$ is a binomial. $(a+b)^n=\underbrace{(a+b)(a+b)(a+b)\ldots(a+b)}_{n \text{ factors}}$

The product of these n factors is called binomial expansion and can be written as follows:

$(a+b)^n={}_nC_0\,a^n+{}_nC_1\,a^{n-1}b+\ldots+{}_nC_{r-1}\,a^{n-r+1}b^{r-1}+\ldots+{}_nC_n\,b^n$,

where the numerical coefficients are the combinations of n objects (b's) taken r at a time.

Look at the following example.

Example 30

Use binomial expansion to expand $(2a-3)^5$.

Solution:

The first term is $(2a)$ and the second term is (-3).

$n=5$

$$(2a-3)^5={}_5C_0(2a)^5+{}_5C_1(2a)^4(-3)^1+{}_5C_2(2a)^3(-3)^2+{}_5C_3(2a)^2(-3)^3+{}_5C_4(2a)^1(-3)^4+{}_5C_5(-3)^5$$

$$=(1)(32a^5)+(5)(16a^4)(-3)+(10)(8a^3)(9)+(10)(4a^2)(-27)+(5)(2a)(81)+(1)(-243)$$

$$=32a^5-240a^4+720a^3-1080a^2+810a-243$$

The rth term in a binomial expansion $(a+b)^n$ is $t_r={}_nC_{r-1}\,a^{n-r+1}b^{r-1}$.

Example 31

Determine the twentieth term in the expansion of $(3a-b)^{25}$.

Solution:

$$t_{20} = {}_{25}C_{20-1}(3a)^{25-20+1}(-b)^{20-1}$$
$$= \frac{25!}{19!6!}(3a)^{6}(-b)^{19}$$
$$= (177\,100)(729)a^{6}(-b)^{19}$$
$$= -129\,105\,900a^{6}b^{19}$$

Blaise Pascal arranged the coefficients in the binomial expansions in the form of a triangular array. This triangle enables you to find the coefficients easily when the exponent (n) of the binomial ($a + b$) is relatively small. Please review Pascal's triangle in the last activity.

Pascal's triangle tells two important ideas:

$${}_nC_r = {}_nC_{n-r}$$

$${}_{n+1}C_{r+1} = {}_nC_r + {}_nC_{r+1}$$

Example 32

Find another expression in the form ${}_nC_r$ which is equal to ${}_{15}C_7$.

Solution:

$${}_nC_r = {}_nC_{n-r}$$
$${}_{15}C_7 = {}_{15}C_8$$
$$\uparrow \quad \uparrow$$
$$7+8=15$$

Example 33

Find another expression in the form ${}_nC_r$ which is equal to ${}_{10}C_6 + {}_{10}C_7$.

Solution:

$${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$$

$${}_{10}C_6 + {}_{10}C_7 = {}_{11}C_7$$

Do the following questions.

1. The school has seven programs available and a student has to choose five of the seven programs. In how many ways can this be done?

2. A student wants to choose three or more programs out of five programs available in school. In how many ways can this be done?

3. In how many ways can two shirts and three ties be selected from five shirts and six ties?

4. There are eleven dishes and five soft drinks. In how many ways can five items be chosen with at least one soft drink?

5. An ordinary deck of 52 cards has four aces. Two cards are drawn at random. What is the probability that both cards are aces?

6. Expand $(x-5y)^4$.

7. Determine the ninth term in the expression of $(x+4y)^{10}$.

8. Find another expression in the form ${}_nC_r$ which is equal to ${}_{28}C_{19}$.

9. Find another expression in the form ${}_nC_r$ which is equal to ${}_{18}C_8+{}_{18}C_9$.

For solutions to **Extra Help**, turn to the **Appendix, Topic 2**.

Extensions

In this section, you are going to learn two things:

- problems involving both permutations and combinations
- other patterns that arise from Pascal's triangle

Problems Involving Both Permutations and Combinations

Some problems involve both permutations and combinations. Those questions are usually more challenging. To solve this type of question, usually you have to calculate the number of combinations first.

Example 34

Given seven consonants and four vowels, how many five-letter arrangements can be made if each arrangement consists of three different consonants and two different vowels?

Solution:

Selections first: The three consonants can be selected in ${}_7C_3$ ways.
The two vowels can be selected in ${}_4C_2$ ways.

Arrangements: The five letters can be arranged in ${}_5P_5$ ways.

The total number of arrangements $= {}_7C_3 \times {}_4C_2 \times {}_5P_5$
$= 35 \times 6 \times 120$
$= 25\,200$ ways

Example 35

There is a seven-place round table and ten people are available. In how many ways can a seating arrangement be made?

Solution:

Selections first: The seven persons can be selected in ${}_{10}C_7$ ways.
Arrangements: The seven-place round table can have $(7-1)!$ arrangements.

The total number of arrangements $= {}_{10}C_7 \times (7-1)!$

$= 120 \times 720$

$= 86\ 400$ arrangements

Other Patterns that Arise from Pascal's Triangle

You are going to investigate further the other patterns that arise from Pascal's triangle.

Investigation 1: Find the sum of the numbers in each row of Pascal's triangle. Write each sum in the blank spaces provided.

	Pascal's Triangle													Sum		2^n	
$(a+b)^0$							1							=	_____	=	_____
$(a+b)^1$						1	+	1						=	_____	=	_____
$(a+b)^2$					1	+	2	+	1					=	_____	=	_____
$(a+b)^3$				1	+	3	+	3	+	1				=	_____	=	_____
$(a+b)^4$			1	+	4	+	6	+	4	+	1			=	_____	=	_____
$(a+b)^5$		1	+	5	+	10	+	10	+	5	+	1		=	_____	=	_____
$(a+b)^6$	1	+	6	+	15	+	20	+	15	+	6	+	1	=	_____	=	_____

The sums of the numbers in the next two rows are _____ and _____.

For solutions to **Investigation 1**, turn to the **Appendix, Topic 2.**

Investigation 2: Find the sum of the squares of the numbers in each row of Pascal's triangle. Write each sum in the blank spaces provided.

	Sum	${}_nC_r$
1^2	= _____	= _____
$1^2 + 1^2$	= _____	= _____
$1^2 + 2^2 + 1^2$	= _____	= _____
$1^2 + 3^2 + 3^2 + 1^2$	= _____	= _____
$1^2 + 4^2 + 6^2 + 4^2 + 1^2$	= _____	= _____
$1^2 + 5^2 + 10^2 + 10^2 + 5^2 + 1^2$	= _____	= _____
$1^2 + 6^2 + 15^2 + 20^2 + 15^2 + 6^2 + 1^2$	= _____	= _____

The sum of the squares of the numbers in the form ${}_nC_r$ in the next two rows are _____ and _____.

For solutions to **Investigation 2**, turn to the **Appendix, Topic 2**.

Investigation 3: Find the sum of the numbers in each row in the expansion of $(a-b)^n$, where $n \in N$. Write each sum in the blank spaces provided.

	Sum
1	= _____
$1 - 1$	= _____
$1 - 2 + 1$	= _____
$1 - 3 + 3 - 1$	= _____
$1 - 4 + 6 - 4 + 1$	= _____
$1 - 5 + 10 - 10 + 5 - 1$	= _____
$1 - 6 + 15 - 20 + 15 - 6 + 1$	= _____

The sum of the numbers in the next two rows are _____ and _____.

For solutions to **Investigation 3**, turn to the **Appendix, Topic 2**.

Do the following questions.

1. There are 26 students in a class. Six students will be selected to form a committee. Two members of the committee will be designated as chairperson and as secretary. In how many ways can this be done?

2. Linda wants to string ten different charms to make a charm bracelet. If there are fifteen different charms available, how many different charm bracelets can be made?

3. In how many ways can four books be arranged on a bookshelf if there are three different chemistry books and two different physics books, and it is required that the two physics books are together and on the shelf?

4. Use the result from your investigations to evaluate $\sum_{r=0}^{12} {}_{12}C_r$.

5. Write another expression in the form ${}_nC_r$ which is equal to each of the following: (Refer to the second investigation.)

   a. $\binom{2}{0}^2+\binom{2}{1}^2+\binom{2}{2}^2$

   b. $\binom{4}{0}^2+\binom{4}{1}^2+\binom{4}{2}^2+\binom{4}{3}^2+\binom{4}{4}^2$

   c. $\binom{7}{0}^2+\binom{7}{1}^2+\binom{7}{2}^2+\binom{7}{3}^2+\binom{7}{4}^2+\binom{7}{5}^2+\binom{7}{6}^2+\binom{7}{7}^2$

6. Use the result from your investigations to evaluate
   ${}_{10}C_0-{}_{10}C_1+{}_{10}C_2-{}_{10}C_3+{}_{10}C_4-{}_{10}C_5+{}_{10}C_6-{}_{10}C_7+{}_{10}C_8-{}_{10}C_9+{}_{10}C_{10}$.

For solutions to **Extensions**, turn to the **Appendix, Topic 2**.

${}_nC_r=\binom{n}{r}$

Unit Summary

What You Have Learned

In this unit you studied the following concepts:

- the fundamental counting principle
- ${}_nP_n$ and $n!$
- ${}_nP_r$ and linear permutations with repetitions
- circular permutations and permutations with reflections
- probability questions which involve the use of permutations
- combinations and the meaning and applications of the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$
- probability questions which involve the use of combinations
- the binomial theorem and Pascal's triangle

You are now ready to complete the **Unit Assignment**.

Appendix

Solutions

Review

Topic 1 Permutations

Topic 2 Combinations

Review

1. $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

2. $\frac{3}{9} \times \frac{5}{12} = \frac{5}{36}$

3. $\frac{7}{19} \times \frac{6}{18} = \frac{7}{57}$

4. $\frac{6}{36} + \frac{6}{36} - \frac{1}{36} = \frac{11}{36}$

5. $\frac{1}{6} + \frac{1}{6} = \frac{1}{3}$

   (mutually exclusive events)

Exploring Topic 1

Activity 1

Solve problems using the fundamental counting principle.

1. a. No, it cannot since the first digits 5, 7, and 8 will produce numbers greater than 400.

   b. The first digit can be chosen in **2** ways.
   A 2 or 3 can be used for the first digit.

   c. The second digit can be chosen in **5** ways.
   Any of the five digits can be used for the second digit since digits can be repeated.

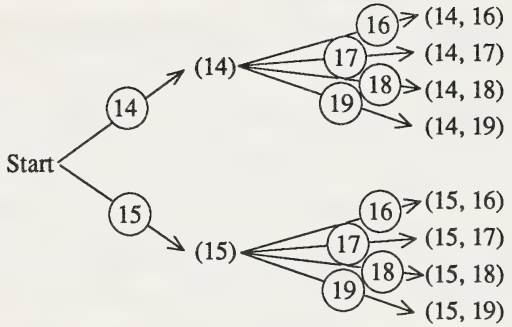
   d. The third digit can be chosen in **5** ways.
   Any of the five digits can be used for the third digit since digits can be repeated.

   e. Apply the fundamental counting principle. The number of three-digit numbers is **50**.
   $(2 \times 5 \times 5)$

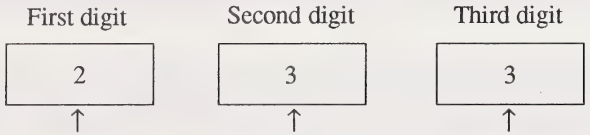
2. a. $8 \times 7 = 56$ ways

   b. $15 \times 15 = 225$ ways

3. Town *A* Village *B* City *C*



4. The following three boxes are used to represent the number of choices for each digit in a three-digit odd number.



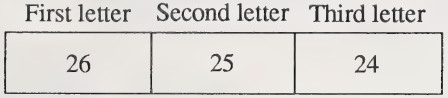
One of 2 or 4 can go here; thus, there are two choices.

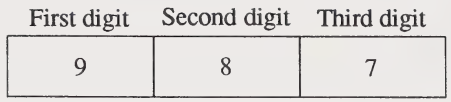
There are three choices for this digit after the choices are made for the first and third digits.

One of 1, 7, or 9 can go here; thus, there are three choices.

Restrictions must be dealt with first when doing these problems. Since the three-digit odd numbers are between 200 and 600, this means that a 2 or a 4 can be used for the first digit. Three-digit odd numbers are required; thus, the third digit must end in a 1, 7, or 9. This means that there are three choices for the third digit. When the choice of numbers has been determined for the first and third digits, there are three choices for the second digit since digits cannot be repeated. For example, if a 4 is used for the first digit and a 7 is used for the third digit, three choices (1, 2, or 9) can be used for the second digit. The total number of three-digit odd numbers = $2 \times 3 \times 3 = 18$.

5. The first three boxes are used to represent the number of choices for the three letters. The next three boxes represent the number of choices for the three digits.





There are 26 choices for the first letter since there are 26 letters in the alphabet. Since letters cannot be repeated like **AAB**, there are 25 choices for the second letter. The third letter has 24 choices since repetitions are not allowed.

There are nine choices for the first digit. Digits cannot be repeated; thus, there are eight choices for the second digit. The third digit has seven choices.

$$\begin{aligned}\text{The number of licence plates} &= (26 \times 25 \times 24) \times (9 \times 8 \times 7) \\ &= 15\,600 \times 504 \\ &= 7\,862\,400\end{aligned}$$

(Note: If both letters and digits could be repeated, the number of licence plates would be $26 \times 26 \times 26 \times 9 \times 9 \times 9 = 12\,812\,904$.)

6. Letter 1 can go into any one of the two mailboxes. Letter 2 can go into any one of the two mailboxes, and so on. Thus, the total number of ways is $2 \times 2 \times 2 \times 2 \times 2 = 32$ ways.

Activity 2

Define and use ${}_nP_n$ and $n!$.

1. $7! = 5040$ ways

2. $5! = 120$ ways

3. $\frac{10!}{7!2!} = \frac{10\times9\times8\times\cancel{7!}}{2\times1\times\cancel{7!}}$

$= 360$

4. $\frac{4!}{5!2!} + \frac{6!}{3!2!} = \frac{1}{5\times2!} + \frac{6\times5\times4}{2\times1}$

$= \frac{1}{10} + 60$

$= 60\frac{1}{10}$ or 60.1

5. $(k-2)(k-3)! = (k-2)!$

6. $\frac{(n+r-2)!}{(n+r-5)!} = \frac{(n+r-2)(n+r-3)(n+r-4)(n+r-5)!}{(n+r-5)!}$

$= (n+r-2)(n+r-3)(n+r-4)$

7. $(m+3)(m+2)! = (m+3)!$

8. $\frac{(n-3)!}{(n-4)!} = 7$

$\frac{(n-3)(n-4)!}{(n-4)!} = 7$

$(n-3) = 7$

$n = 10$

9. The two vowels are **U** and **E**. Thus, there are two choices for the first letter.

2	5	4	3	2	1

$2\times5\times4\times3\times2\times1 = 240$ words

Activity 3

Define and use ${}_nP_r$; also, solve linear permutations with repetitions and permutations with like elements.

1. ${}_{10}P_4 = \frac{10!}{(10-4)!}$

$= \frac{10\times9\times8\times7\times6!}{6!}$

$= 5040$ ways

2. ${}_4P_3 + {}_5P_2 = \frac{4!}{(4-3)!} + \frac{5!}{(5-2)!}$

$= 4! + 5\times4$

$= 44$

3. The numbers can contain four digits, three digits, two digits, or one digit.

${}_5P_4 + {}_5P_3 + {}_5P_2 + {}_5P_1 = \frac{5!}{(5-4)!} + \frac{5!}{(5-3)!} + \frac{5!}{(5-2)!} + \frac{5!}{(5-1)!}$

$= 120 + 60 + 20 + 5$

$= 205$

4. $\left(\frac{1}{2}\right){}_nP_7 = {}_{n-2}P_5,\ n \in N$

$$\left(\frac{1}{2}\right)\frac{n!}{(n-7)!} = \frac{(n-2)!}{(n-2-5)!}$$

$$\frac{n!}{2(n-7)!} = \frac{(n-2)!}{(n-7)!}$$

$$\frac{n!}{2} = (n-2)!$$

$$\frac{n!}{(n-2)!} = 2$$

$$n(n-1) = 2$$

$$n^2 - n - 2 = 0$$

$$(n-2)(n+1) = 0$$

$n = 2,\ -1$ (−1 is not a natural number.)

$\therefore n = 2$

5. The two physics books can be arranged in 2! ways. Consider the physics books as one group since they must be together. The three chemistry books plus this one physics group totals four groups. These can be arranged in ${}_4P_4$ ways.

${}_2P_2 \times {}_4P_4 = 2! \times 4!$

$= 48$ ways

6. There are three E's and two S's in the word PEERLESS.

$$\frac{8!}{3!2!}$$

$$= \frac{8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1}{3 \times 2 \times 1 \times 2 \times 1}$$

$= 3360$ permutations

7. $26 \times 26 \times 10 \times 10 \times 10 \times 10 = 6\ 760\ 000$

8. The marbles and beads may each be arranged in ${}_4P_4$ ways. The order of marbles and beads may be reversed; thus, multiply by 2.

The total number of arrangements $= {}_4P_4 \times {}_4P_4 \times 2$

$= 4! \times 4! \times 2$

$= 1152$

9. The two different biology books can be arranged among themselves in 2! ways.

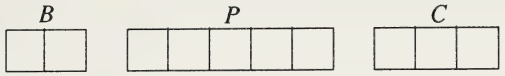
The three different chemistry books can be arranged among themselves in 3! ways.

The five different physics books can be arranged among themselves in 5! ways.

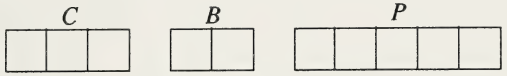
The three different groups of books can be arranged amongst themselves in 3! ways as shown in the diagrams that follow. *B* represents the biology books, *C* represents the chemistry books, and *P* represents the physics books.

Group Arrangement Order

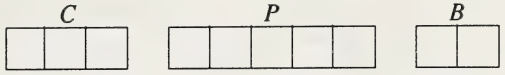
BCP



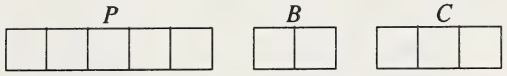
BPC



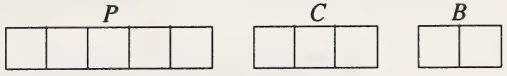
CBP



CPB



PBC



PCB

3! = 6 group arrangements

The total number of arrangements $= 2! \times 3! \times 5! \times 3!$

$= 2 \times 6 \times 120 \times 6$

$= 8640$

10. $\frac{8!}{3!5!}$

$= \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$

$= 56$ ways

Activity 4

Solve problems involving circular permutations and permutations with reflections.

1. $(9-1)! = 8!$
   $= 40\ 320$ ways

2. $\frac{(12-1)!}{2!} = \frac{11!}{2!}$
   $= 19\ 958\ 400$ ways

3. Assume the girls are seated first. A position is assigned to a girl and the other four girls are arranged in 4! ways. The five boys are arranged in 5! ways. The total number of circular arrangements is $(5 - 1)! \times 5! = 24 \times 120 = 2880$ ways.

4. The three marbles must be adjacent; thus, consider the three marbles as one group of marbles. These three marbles can be arranged in 3! ways. The remaining four marbles plus this one group of marbles gives five marbles. These five marbles can be arranged in $(5 - 1)!$ ways. The total number of arrangements is $3! \times (5 - 1)! = 6 \times 24 = 144$ ways.

5. $\frac{(5-1)!}{2} = \frac{4!}{2}$
   $= 12$ ways

6. $\frac{(9-1)!}{4!5!2} = \frac{8!}{4!5!2}$
   $= \frac{8 \times 7 \times 6}{4 \times 3 \times 2 \times 2}$
   $= 7$ ways

Activity 5

Solve probability questions which involve the use of permutations.

1. $P(E) = \frac{1}{{}_7P_7}$
   $= \frac{1}{7!}$
   $= \frac{1}{5040}$

2. $\frac{{}_3P_3}{{}_8P_3} = \frac{3!}{\frac{8!}{(8-3)!}}$
   $= \frac{3!5!}{8!}$
   $= \frac{3 \times 2 \times 1}{8 \times 7 \times 6}$
   $= \frac{1}{56}$

3. There are two black kings, two red queens, and two black jacks.
The number of favourable cases $= 2 \times 2 \times 2$
The number of possible cases $= {}_{52}P_3$

$$\therefore \text{Probability} = \frac{2 \times 2 \times 2}{{}_{52}P_3}$$
$$= \frac{8}{52 \times 51 \times 50}$$
$$= \frac{1}{16\,575}$$

Extra Help

1. ${}_5P_4 = \frac{5!}{(5-4)!}$
$= 5!$
$= 120$

2. Any four letters $= {}_4P_4$
Any three letters $= {}_4P_3$
Any two letters $= {}_4P_2$
Any one letter $= {}_4P_1$
Total $= {}_4P_4 + {}_4P_3 + {}_4P_2 + {}_4P_1$
$= 4! + 4! + 12 + 4$
$= 24 + 24 + 12 + 4$
$= 64$

3. Two boys can be seated in ${}_2P_2 = 2!$ ways.
The two boys always occupy the first two chairs. The three girls can be seated in ${}_3P_3$ ways.
Total $= 2! \times {}_3P_3$
$= 2! \times 3!$
$= 12$ ways

4. There are eight beads. They can be arranged in ${}_8P_8 = 8!$ ways.
Five beads are red and three are blue.

The number of arrangements $= \frac{8!}{5!3!}$
$= \frac{8 \times 7 \times 6}{3 \times 2 \times 1}$
$= 56$

5. $(10-1)!$
$= 9!$
$= 362\,880$ ways

6. $\frac{(7-1)!}{2} = \frac{6!}{2}$
$= 360$ ways

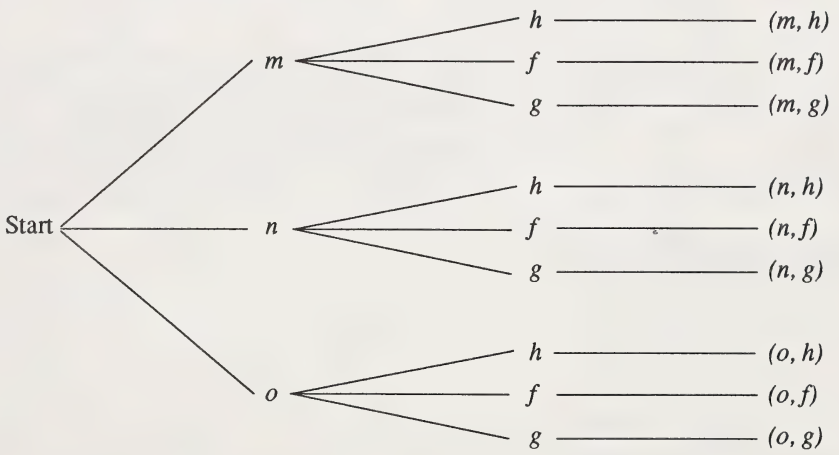
7. There are thirteen clubs and two red kings in a deck of cards.
The number of favourable cases = 13×2
The number of possible cases = ${}_{52}P_2$

$$= \frac{52!}{50!}$$
$$= 52 \times 51$$
$$= 2652$$

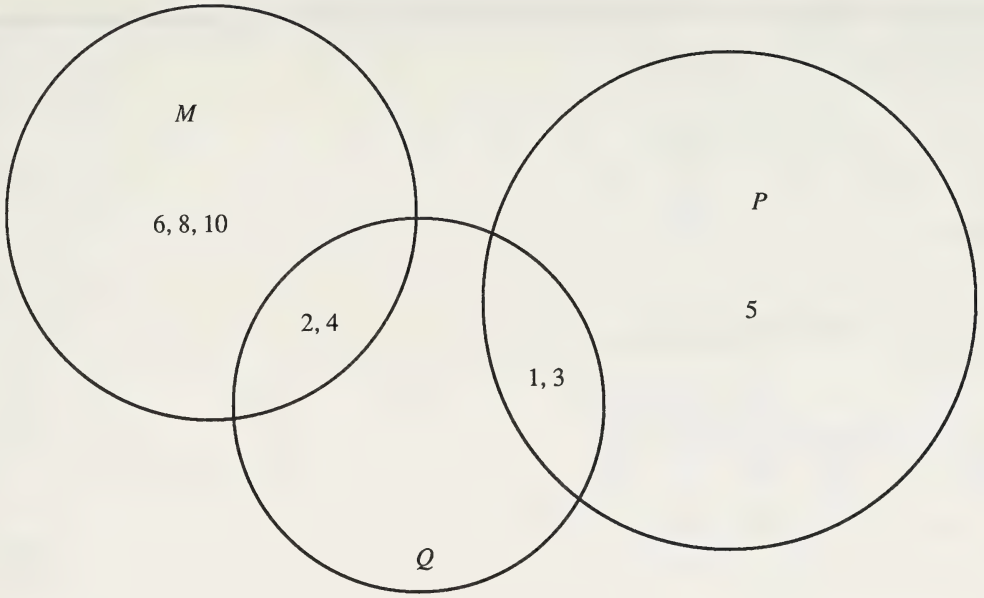
$$\text{Probability} = \frac{13 \times 2}{2652}$$
$$= \frac{1}{102}$$

Extensions

1.



2.



$n(P \cap Q) = 2$

$n(M \cap P) = 0$

$n(M \cup P) = 8$

$n(P \cup Q) = 5$

Exploring Topic 2

Activity 1

Define the meaning and use applications of the formula ${}_nC_r = \frac{n!}{(n-r)!r!}$.

1. ${}_{17}C_3 = \frac{17!}{(17-3)!3!}$

$= \frac{17!}{14!3!}$

$= \frac{17\times16\times15}{3\times2\times1}$

$= 680$

2. ${}_4C_3 + {}_5C_0 - {}_6C_1$

$= \frac{4!}{3!1!} + \frac{5!}{0!5!} - \frac{6!}{5!1!}$

$= 4 + 1 - 6$

$= -1$

3. LS $= {}_6C_3 + {}_6C_4$

$= \frac{6!}{3!3!} + \frac{6!}{4!2!}$

$= \frac{6\times5\times4}{3\times2\times1} + \frac{6\times5}{2\times1}$

$= 20 + 15$

$= 35$

RS $= {}_7C_4$

$= \frac{7!}{4!3!}$

$= \frac{7\times6\times5}{3\times2}$

$= 35$

$\therefore {}_6C_3 + {}_6C_4 = {}_7C_4$

4. ${}_{10}C_7 + {}_{10}C_8 + {}_{10}C_9 + {}_{10}C_{10}$

$= \frac{10!}{7!3!} + \frac{10!}{8!2!} + \frac{10!}{9!1!} + \frac{10!}{10!0!}$

$= \frac{10\times9\times8}{3\times2\times1} + \frac{10\times9}{2\times1} + 10 + 1$

$= 120 + 45 + 10 + 1$

$= 176$ ways

5. $\frac{{}_{10}C_5}{2} = \frac{\frac{10!}{5!5!}}{2}$

$= \frac{10\times9\times8\times7\times6}{5\times4\times3\times2\times2}$

$= 126$ ways

6. ${}_3C_1 \times {}_8C_2 \times {}_{10}C_3 \times {}_9C_4$

$= \frac{3!}{2!1!} \times \frac{8!}{6!2!} \times \frac{10!}{7!3!} \times \frac{9!}{5!4!}$

$= 3 \times 28 \times 120 \times 126$

$= 1\,270\,080$ ways

7. ${}_{12}C_3 + {}_{12}C_2 + {}_{12}C_1$

$= \frac{12!}{9!3!} + \frac{12!}{10!2!} + \frac{12!}{11!1!}$

$= 220 + 66 + 12$

$= 298$ ways

8. ${}_{21}C_9 \times {}_{12}C_7 \times {}_5C_5$

$= \frac{21!}{9!12!} \times \frac{12!}{7!5!} \times 1$

$= 293\,930 \times 792 \times 1$

$= 232\,792\,560$ ways

9. $2^{10} - 1 = 1023$ ways

10. $2^8 = 256$ ways

11. a. ${}_{40}C_5 = 658\,008$ ways

b. ${}_{12}C_2 \times {}_{40}C_3 = \frac{12 \times 11}{1 \times 2} \times \frac{40 \times 39 \times 38}{3 \times 2 \times 1}$

$= 652\,080$ ways

c. The kings are always excluded. There are 48 cards left. The two red queens are always included. The other three cards must be selected from the $(48 - 2) = 46$ cards left.

${}_2C_2 \times {}_{46}C_3 = 1 \times \frac{46!}{43!3!}$

$= 15\,180$ ways

d. The number of any five-card hands $= {}_{52}C_5$

$= 2\,598\,960$

No king is included $= {}_{48}C_5$

$= 1\,712\,304$

At least one king is included $= 2\,598\,960 - 1\,712\,304$

$= 886\,656$ ways

12. The number of straight lines formed by joining any two of twelve points $= {}_{12}C_2$

$= 66$

The number of straight lines formed by joining any two of four points $= {}_4C_2$

$= 6$

Since four of the twelve points form one straight line, the number of straight lines formed by joining these twelve points $= 66 - 6 + 1$

$= 61$

13. ${}_{14}C_2 - 14$

$= \frac{14!}{12!2!} - 14$

$= 91 - 14$

$= 77$

14. The number of ways to choose one or more students $= 2^{15} - 1$
The number of ways to choose one or two students
$= {}_{15}C_1 + {}_{15}C_2$
The number of ways to choose three or more
students $= \left(2^{15} - 1\right) - \left({}_{15}C_1 + {}_{15}C_2\right)$

$= 32\,767 - 15 - 105$

$= 32\,647$ ways

15. The number of ways to choose at least one shirt $= 2^5 - 1$
The number of ways to choose at least one pair of dress pants
$= 2^6 - 1$
The number of ways to choose any number of hats $= 2^3$
The number of ways to choose the three items

$= \left(2^5 - 1\right)\left(2^6 - 1\right)\left(2^3\right)$

$= (31)(63)(8)$

$= 15\,624$ ways

16. ${}_9C_2 = \frac{9!}{7!2!}$

$= \frac{9 \times 8}{2 \times 1}$

$= 36$ games

Activity 2

Solve probability questions which involve the use of combinations.

1. The number of ways to choose any two cards $= {}_9C_2$
The number of ways to choose any two even numbers $= {}_4C_2$
The probability $P(E) = \frac{{}_4C_2}{{}_9C_2}$

$= \frac{\frac{4!}{2!2!}}{\frac{9!}{7!2!}}$

$= \frac{6}{36}$

$= \frac{1}{6}$

2. There is only one way of buying two winning tickets. Thus, the probability of buying two winning tickets is $\frac{1}{{}_{50}C_2}$.

The probability of buying no winning tickets is $\frac{{}_{48}C_2}{{}_{50}C_2}$.

The probability of buying exactly one winning ticket

$= 1 - \frac{{}_{48}C_2}{{}_{50}C_2} - \frac{1}{{}_{50}C_2}$

$= 1 - \frac{1128}{1225} - \frac{1}{1225}$

$= \frac{96}{1225}$

Another method:

The number of ways to choose exactly one winning ticket

$= {}_2C_1 \times {}_{48}C_1$

$= 2 \times 48$

$= 96$

The number of ways to choose any two tickets $= {}_{50}C_2$

$= 1225$

$$\text{Probability} = \frac{96}{1225}$$

3. The number of favourable cases = $1 \times 3 = 3$

The number of possible cases $= {}_2C_1 \times {}_6C_1$

$= 12$

$$\text{Probability } P(E) = \frac{3}{12}$$

$$= \frac{1}{4}$$

4. $$\text{The probability of drawing two even numbers} = \frac{{}_6C_2}{{}_{12}C_2}$$

$$\text{The probability of drawing two numbers greater than 5} = \frac{{}_7C_2}{{}_{12}C_2}$$

The probability that both numbers are even or both are greater than 5

$$= \frac{{}_6C_2}{{}_{12}C_2} + \frac{{}_7C_2}{{}_{12}C_2}$$

$$= \frac{15}{66} + \frac{21}{66}$$

$$= \frac{36}{66}$$

$$= \frac{6}{11}$$

5. The number of ways that a committee of four can be chosen $= {}_{25}C_4$

The number of ways that a committee of four boys can be chosen $= {}_{15}C_4$

$$\text{The probability that at least one girl is chosen} = 1 - \frac{{}_{15}C_4}{{}_{25}C_4}$$

$$= 1 - \frac{\frac{15!}{11!4!}}{\frac{25!}{21!4!}}$$

$$= 1 - \frac{1365}{12\,650}$$

$$= \frac{11\,285}{12\,650}$$

$$= \frac{2257}{2530}$$

Activity 3

Define and use the binomial theorem and Pascal's triangle.

1. a. $(x+2y)^8 = {}_8C_0x^8 + {}_8C_1x^7(2y) + {}_8C_2x^6(2y)^2 + {}_8C_3x^5(2y)^3 + {}_8C_4x^4(2y)^4 + {}_8C_5x^3(2y)^5 + {}_8C_6x^2(2y)^6 + {}_8C_7x(2y)^7 + {}_8C_8(2y)^8$

$= x^8 + 8x^7(2y) + 28x^6(2y)^2 + 56x^5(2y)^3 + 70x^4(2y)^4 + 56x^3(2y)^5 + 28x^2(2y)^6 + 8x(2y)^7 + (2y)^8$

$= x^8 + 16x^7y + 112x^6y^2 + 448x^5y^3 + 1120x^4y^4 + 1792x^3y^5 + 1792x^2y^6 + 1024xy^7 + 256y^8$

b. $(2x-y)^7 = {}_7C_0(2x)^7 + {}_7C_1(2x)^6(-y) + {}_7C_2(2x)^5(-y)^2 + {}_7C_3(2x)^4(-y)^3 + {}_7C_4(2x)^3(-y)^4 + {}_7C_5(2x)^2(-y)^5 + {}_7C_6(2x)(-y)^6 + {}_7C_7(-y)^7$

$= 128x^7 + 7(64x^6)(-y) + (21)(32x^5)(y^2) + (35)(16x^4)(-y^3) + (35)(8x^3)(y^4) + 21(4x^2)(-y^5) + 7(2x)(y^6) - y^7$

$= 128x^7 - 448x^6y + 672x^5y^2 - 560x^4y^3 + 280x^3y^4 - 84x^2y^5 + 14xy^6 - y^7$

c. $\left(x^2 - \frac{1}{2}y\right)^5 = {}_5C_0\left(x^2\right)^5 + {}_5C_1\left(x^2\right)^4\left(-\frac{1}{2}y\right) + {}_5C_2\left(x^2\right)^3\left(-\frac{1}{2}y\right)^2 + {}_5C_3\left(x^2\right)^2\left(-\frac{1}{2}y\right)^3 + {}_5C_4\left(x^2\right)\left(-\frac{1}{2}y\right)^4 + {}_5C_5\left(-\frac{1}{2}y\right)^5$

$= x^{10} + (5)\left(x^8\right)\left(-\frac{1}{2}y\right) + (10)\left(x^6\right)\left(-\frac{1}{2}y\right)^2 + (10)\left(x^4\right)\left(-\frac{1}{2}y\right)^3 + (5)\left(x^2\right)\left(-\frac{1}{2}y\right)^4 + \left(-\frac{1}{2}y\right)^5$

$= x^{10} - \frac{5}{2}x^8y + \frac{5}{2}x^6y^2 - \frac{5}{4}x^4y^3 + \frac{5}{16}x^2y^4 - \frac{1}{32}y^5$

2. The rth term in $(a+b)^n = {}_nC_{r-1}a^{n-r+1}b^{r-1}$

The ninth term $= {}_{15}C_8x^{15-9+1}(-2)^{9-1}$

$= \frac{15!}{8!7!}x^7(-2)^8$

$= 6435x^7(256)$

$= 1\,647\,360x^7$

3. The middle term is the fourth term since there are seven terms in the expansion of $\left(x^2-\sqrt{y}\right)^6$. Thus, $r=4$.
The rth term is ${}_nC_{r-1}a^{n-r+1}b^{r-1}$.

$$t_4={}_6C_3\left(x^2\right)^{6-4+1}\left(-\sqrt{y}\right)^{4-1}$$
$$=\frac{6!}{3!3!}\left(x^2\right)^3\left(-\sqrt{y}\right)^3$$
$$=-20x^6y^{\frac{3}{2}} \text{ or } -20x^6\sqrt{y^3}$$

4. The rth term is ${}_nC_{r-1}a^{n-r+1}b^{r-1}$. When the literal coefficient of a term is a^8b^8 in the expansion of $(a-b)^{16}$, then $n=16$ and $r-1=8$.

The numerical coefficient $={}_nC_{r-1}$
$={}_{16}C_8$

5. In the expansion $(x+y)^n$, the sum of the exponents for any one term in the expansion is always equal to the value n.

$x+7=12$
$x=5$

6. The rth term is ${}_nC_{r-1}a^{n-r+1}b^{r-1}$.

$r-1=2$
$r=3$

Thus, the third term is $\frac{15}{16}x^4y^2$.

7. ${}_nC_r={}_nC_{n-r}$
${}_{14}C_6={}_{14}C_{14-6}$
$={}_{14}C_8$

8. Pascal's theorem: ${}_nC_r+{}_nC_{r+1}={}_{n+1}C_{r+1}$
${}_{17}C_9+{}_{17}C_{10}={}_{18}C_{10}$

9. $(a+b-c)^4=\left[a+(b-c)\right]^4$
$={}_4C_0a^4+{}_4C_1a^3(b-c)+{}_4C_2a^2(b-c)^2$
$+{}_4C_3a(b-c)^3+{}_4C_4(b-c)^4$
$=a^4+4a^3(b-c)+6a^2(b-c)^2+4a(b-c)^3$
$+(b-c)^4$
$=a^4+4a^3b-4a^3c+6a^2\left(b^2-2bc+c^2\right)$
$+4a\left(b^3-3b^2c+3bc^2-c^3\right)$
$+\left(b^4-4b^3c+6b^2c^2-4bc^3+c^4\right)$
$=a^4+4a^3b-4a^3c+6a^2b^2-12a^2bc+6a^2c^2$
$+4ab^3-12ab^2c+12abc^2-4ac^3+b^4$
$-4b^3c+6b^2c^2-4bc^3+c^4$

10. $2^7=(1+1)^7$
$={}_7C_0(1)^7+{}_7C_1(1)^6(1)+{}_7C_2(1)^5(1)^2+{}_7C_3(1)^4(1)^3$
$+{}_7C_4(1)^3(1)^4+{}_7C_5(1)^2(1)^5+{}_7C_6(1)(1)^6+{}_7C_7(1)^7$
$={}_7C_0+{}_7C_1+{}_7C_2+{}_7C_3+{}_7C_4+{}_7C_5+{}_7C_6+{}_7C_7$

Extra Help

1. $${}_7C_5 = \frac{7!}{2!5!}$$
$$= \frac{7\times 6}{2\times 1}$$
$$= 21$$

2. $${}_5C_3 + {}_5C_4 + {}_5C_5 = \frac{5!}{3!2!} + \frac{5!}{4!1!} + \frac{5!}{5!0!}$$
$$= 10 + 5 + 1$$
$$= 16$$

3. $${}_5C_2 \times {}_6C_3 = \frac{5!}{2!3!} \times \frac{6!}{3!3!}$$
$$= 10 \times 20$$
$$= 200 \text{ ways}$$

4. With no soft drink, all five dishes $= {}_{11}C_5$
$$= \frac{11!}{5!6!}$$
$$= 462$$

Any five items $= {}_{16}C_5$
$$= \frac{16!}{5!11!}$$
$$= 4368$$

The number of ways five items can be chosen with at least one soft drink $= {}_{16}C_5 - {}_{11}C_5$
$$= 4368 - 462$$
$$= 3906 \text{ ways}$$

5. Number of favourable cases $= {}_4C_2 = \frac{4!}{2!2!}$
$$= 6$$

Number of possible cases $= {}_{52}C_2 = \frac{52!}{50!2!}$
$$= \frac{52\times 51}{2\times 1}$$
$$= 1326$$

Probability $P(E) = \frac{6}{1326}$
$$= \frac{1}{221}$$

6. $$(x-5y)^4 = {}_4C_0x^4 + {}_4C_1x^3(-5y) + {}_4C_2x^2(-5y)^2 + {}_4C_3x(-5y)^3 + {}_4C_4x^0(-5y)^4$$
$$= x^4 - 20x^3y + 150x^2y^2 - 500xy^3 + 625y^4$$

7. rth term $= {}_nC_{r-1}a^{n-r+1}b^{r-1}$

$$t_9 = {}_{10}C_{9-1}x^{10-9+1}(4y)^{9-1}$$
$$= \frac{10!}{8!2!}x^2(4y)^8$$
$$= 45(x^2)(65\,536)y^8$$
$$= 2\,949\,120x^2y^8$$

8. $${}_nC_r = {}_nC_{n-r}$$
$${}_{28}C_{19} = {}_{28}C_{28-19}$$
$$= {}_{28}C_9$$

9. ${}_nC_r + {}_nC_{r+1} = {}_{n+1}C_{r+1}$

${}_{18}C_8 + {}_{18}C_9 = {}_{19}C_9$

Extensions

Investigation 1

													Sum		2^n
						1							= 1	=	2^0
					1	+	1						= 2	=	2^1
				1	+	2	+	1					= 4	=	2^2
			1	+	3	+	3	+	1				= 8	=	2^3
		1	+	4	+	6	+	4	+	1			= 16	=	2^4
	1	+	5	+	10	+	10	+	5	+	1		= 32	=	2^5
1	+	6	+	15	+	20	+	15	+	6	+	1	= 64	=	2^6

The sums of the numbers in the next two rows are **128** and **256**.

Investigation 2

													Sum		${}_nC_r$
						1^2							= 1	=	${}_2C_0$
					1^2	+	1^2						= 2	=	${}_2C_1$
				1^2	+	2^2	+	1^2					= 6	=	${}_4C_2$
			1^2	+	3^2	+	3^2	+	1^2				= 20	=	${}_6C_3$
		1^2	+	4^2	+	6^2	+	4^2	+	1^2			= 70	=	${}_8C_4$
	1^2	+	5^2	+	10^2	+	10^2	+	5^2	+	1^2		= 252	=	${}_{10}C_5$
1^2	+	6^2	+	15^2	+	20^2	+	15^2	+	6^2	+	1^2	= 924	=	${}_{12}C_6$

The sum of the squares of the numbers in the form ${}_nC_r$ in the next two rows are ${}_{14}C_7$ and ${}_{16}C_8$.

Investigation 3

													Sum
						1					=	1
				1	–	1					=	0
			1	–	2	+	1				=	0
		1	–	3	+	3	–	1			=	0
	1	–	4	+	6	–	4	+	1		=	0
1	–	5	+	10	–	10	+	5	–	1	=	0
1	–	6	+	15	–	20	+	15	–	6	+	1	=	0

The sum of the numbers in the next two rows are **0** and **0**.

1. The number of ways to form a committee $= {}_{26}C_6$
   The number of ways to choose a chairperson and a secretary $= {}_6P_2$
   The total number of ways
   $= {}_{26}C_6 \times {}_6P_2$
   $= \frac{26!}{6!20!} \times \frac{6!}{4!}$
   $= \frac{26\times25\times24\times23\times22\times21}{6\times5\times4\times3\times2\times1} \times \frac{6\times5\times4\times3\times2\times1}{4\times3\times2\times1}$
   $= 6\ 906\ 900$ ways

2. To choose ten different charms, there are ${}_{15}C_{10}$ ways.
   Number of bracelets formed by stringing ten different charms $= \frac{{}_9P_9}{2}$
   $= \frac{9!}{2}$
   Total number of bracelets $= {}_{15}C_{10} \times \frac{9!}{2}$
   $= \frac{15!}{5!10!} \times \frac{9!}{2}$
   $= \frac{15\times14\times13\times12\times11}{1} \times \frac{9\times8\times7\times6}{2}$
   $= 544\ 864\ 320$ bracelets

3. The two physics books can be arranged in ${}_2P_2 = 2!$ ways.
   The two chemistry books can be selected in ${}_3C_2$ ways.
   The four books can be arranged in ${}_3P_3$ ways. The two physics books are considered as one book since they must be together.

   The total number of ways to select four books and arrange them on a bookshelf $= {}_2P_2 \times {}_3C_2 \times {}_3P_3$
   $= 2! \times \frac{3!}{2!} \times 3!$
   $= 36$ ways

4. $\sum_{r=0}^{12} {}_{12}C_r = {}_{12}C_0 + {}_{12}C_1 + \ldots + {}_{12}C_{12}$
   $= 2^{12}$
   $= 4096$

5. a. $\binom{2}{0}^2 + \binom{2}{1}^2 + \binom{2}{2}^2$
   $= 1^2 + 2^2 + 1^2$
   $= 6$
   $= \binom{4}{2}$

b. $\binom{4}{0}^2+\binom{4}{1}^2+\binom{4}{2}^2+\binom{4}{3}^2+\binom{4}{4}^2$

$=1^2+4^2+6^2+4^2+1^2$

$=70$

$=\binom{8}{4}$

c. $\binom{7}{0}^2+\binom{7}{1}^2+\binom{7}{2}^2+\binom{7}{3}^2+\binom{7}{4}^2+\binom{7}{5}^2+\binom{7}{6}^2+\binom{7}{7}^2$

$=1^2+7^2+21^2+35^2+35^2+21^2+7^2+1^2$

$=3432$

$=\binom{14}{7}$

6. ${}_{10}C_0-{}_{10}C_1+{}_{10}C_2-{}_{10}C_3+{}_{10}C_4-{}_{10}C_5+{}_{10}C_6-{}_{10}C_7+{}_{10}C_8-{}_{10}C_9+{}_{10}C_{10}=0$

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